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Program Verification

Part 3 – Semantics of Functional Programs

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Overview

- definition of a small functional programming language
- operational semantics
- a model in many-sorted logic

Functional Programming – Data Types

Data Type Definitions

- a functional program contains a sequence of data type definitions
- while processing the sequence, we determine the set of types $\mathcal{T}y$, the signature Σ, and the predicates $\mathcal{P}$, which are all initially empty
- each data type definition has the following form

$$\begin{aligned} \mathsf{data}\ \tau = {} & c_1 : \tau_{1,1} \times \ldots \times \tau_{1,m_1} \to \tau \\ \mid {} & \ldots \\ \mid {} & c_n : \tau_{n,1} \times \ldots \times \tau_{n,m_n} \to \tau \end{aligned}$$

where

  - $\tau \notin \mathcal{T}y$ — fresh type name
  - $c_1, \ldots, c_n \notin \Sigma$ and $c_i \neq c_j$ for $i \neq j$ — fresh and distinct constructor names
  - each $\tau_{i,j} \in \{\tau\} \cup \mathcal{T}y$ — only known types
  - exists c_i such that $\tau_{i,j} \in \mathcal{T}y$ for all j — non-recursive constructor
- effect: add type, constructors and equality predicate
  - $\mathcal{T}y := \mathcal{T}y \cup \{\tau\}$
  - $\Sigma := \Sigma \cup \{c_1 : \tau_{1,1} \times \ldots \times \tau_{1,m_1} \to \tau, \ldots, c_n : \tau_{n,1} \times \ldots \times \tau_{n,m_n} \to \tau\}$
  - $\mathcal{P} := \mathcal{P} \cup \{=_\tau \subseteq \tau \times \tau\}$

Data Type Definitions: Examples

- $\mathcal{T}y = \Sigma = \mathcal{P} = \varnothing$
- $\mathsf{data\ Nat} = \mathsf{Zero} : \mathsf{Nat} \mid \mathsf{Succ} : \mathsf{Nat} \to \mathsf{Nat}$
- processing updates $\mathcal{T}y = \{\mathsf{Nat}\}$,
  $\Sigma = \{\mathsf{Zero} : \mathsf{Nat}, \mathsf{Succ} : \mathsf{Nat} \to \mathsf{Nat}\}$
  and $\mathcal{P} = \{=_{\mathsf{Nat}} \subseteq \mathsf{Nat} \times \mathsf{Nat}\}$
- $\mathsf{data\ List} = \mathsf{Nil} : \mathsf{List} \mid \mathsf{Cons} : \mathsf{Nat} \times \mathsf{List} \to \mathsf{List}$
- processing updates $\mathcal{T}y = \{\mathsf{Nat}, \mathsf{List}\}$,
  $\Sigma = \{\mathsf{Zero} : \mathsf{Nat}, \mathsf{Succ} : \mathsf{Nat} \to \mathsf{Nat}, \mathsf{Nil} : \mathsf{List}, \mathsf{Cons} : \mathsf{Nat} \times \mathsf{List} \to \mathsf{List}\}$
  and $\mathcal{P} = \{=_{\mathsf{Nat}} \subseteq \mathsf{Nat} \times \mathsf{Nat}, =_{\mathsf{List}} \subseteq \mathsf{List} \times \mathsf{List}\}$
- $\mathsf{data\ BList} = \mathsf{NilB} : \mathsf{BList} \mid \mathsf{ConsB} : \mathsf{Bool} \times \mathsf{BList} \to \mathsf{BList}$
  not allowed, since $\mathsf{Bool} \notin \mathcal{T}y$
- $\mathsf{data\ LList} = \mathsf{Nil} : \mathsf{LList} \mid \mathsf{Cons} : \mathsf{List} \times \mathsf{LList} \to \mathsf{LList}$
  not allowed, since Nil and Cons are already in Σ
- $\mathsf{data\ Tree} = \mathsf{Node} : \mathsf{Tree} \times \mathsf{Nat} \times \mathsf{Tree} \to \mathsf{Tree}$
  not allowed, since all constructors are recursive

Data Type Definitions: Standard Model

- while processing data type definitions we also build a model $\mathcal{M}$ for the functional program, called the standard model
- when processing

$$\begin{aligned}\textsf{data}\ \tau = {} & c_1 : \tau_{1,1} \times \ldots \times \tau_{1,m_1} \to \tau \\ \mid {} & \ldots \\ \mid {} & c_n : \tau_{n,1} \times \ldots \times \tau_{n,m_n} \to \tau\end{aligned}$$

  - define universe $\mathcal{A}_\tau$ for new type τ inductively via the following inference rules (one for each $1 \leq i \leq n$)

$$\frac{t_1 \in \mathcal{A}_{\tau_{i,1}} \quad \ldots \quad t_{m_i} \in \mathcal{A}_{\tau_{i,m_i}}}{c_i(t_1, \ldots, t_{m_i}) \in \mathcal{A}_\tau}$$

  - define $c_i^{\mathcal{M}}(t_1, \ldots, t_{m_i}) = c_i(t_1, \ldots, t_{m_i})$ uninterpreted constructors
  - define $=_\tau^{\mathcal{M}} = \{(t, t) \mid t \in \mathcal{A}_\tau\}$ equality

Data Type Definitions: Example and Standard Model

- data $\mathsf{Nat} = \mathsf{Zero} : \mathsf{Nat} \mid \mathsf{Succ} : \mathsf{Nat} \to \mathsf{Nat}$
- processing creates universe $\mathcal{A}_{\mathsf{Nat}}$ via the inference rules

$$\frac{}{\mathsf{Zero} \in \mathcal{A}_{\mathsf{Nat}}} \qquad \frac{t \in \mathcal{A}_{\mathsf{Nat}}}{\mathsf{Succ}(t) \in \mathcal{A}_{\mathsf{Nat}}}$$

  i.e., $\mathcal{A}_{\mathsf{Nat}} = \{\mathsf{Zero}, \mathsf{Succ}(\mathsf{Zero}), \mathsf{Succ}(\mathsf{Succ}(\mathsf{Zero})), \dots\}$
- $\mathsf{Zero}^{\mathcal{M}} = \mathsf{Zero} \qquad \mathsf{Succ}^{\mathcal{M}}(t) = \mathsf{Succ}(t)$
- $=^{\mathcal{M}}_{\mathsf{Nat}} = \{(\mathsf{Zero}, \mathsf{Zero}), (\mathsf{Succ}(\mathsf{Zero}), \mathsf{Succ}(\mathsf{Zero})), \dots\}$
- data $\mathsf{List} = \mathsf{Nil} : \mathsf{List} \mid \mathsf{Cons} : \mathsf{Nat} \times \mathsf{List} \to \mathsf{List}$
- processing creates universe $\mathcal{A}_{\mathsf{List}}$ via the inference rules

$$\frac{}{\mathsf{Nil} \in \mathcal{A}_{\mathsf{List}}} \qquad \frac{t_1 \in \mathcal{A}_{\mathsf{Nat}} \quad t_2 \in \mathcal{A}_{\mathsf{List}}}{\mathsf{Cons}(t_1, t_2) \in \mathcal{A}_{\mathsf{List}}}$$

  i.e., $\mathcal{A}_{\mathsf{List}} = \{\mathsf{Nil}, \mathsf{Cons}(\mathsf{Zero}, \mathsf{Nil}), \mathsf{Cons}(\mathsf{Succ}(\mathsf{Zero}), \mathsf{Nil}), \dots\}$
- $=^{\mathcal{M}}_{\mathsf{List}} = \{(\mathsf{Nil}, \mathsf{Nil}), (\mathsf{Cons}(\mathsf{Zero}, \mathsf{Nil}), \mathsf{Cons}(\mathsf{Zero}, \mathsf{Nil})), \dots\}$

Well-Definedness of Standard Model

- question: is the standard model really a model in the sense of many-sorted logic
  - is there a unique type for each $c_i \in \Sigma$ and $=_\tau \in \mathcal{P}$
  - are the definitions of $c_i^\mathcal{M}$ and $=_\tau^\mathcal{M}$ well-defined
  - are the definitions of $\mathcal{A}_\tau$ well-defined, i.e., $\mathcal{A}_\tau \neq \varnothing$
- recall: each data definition has the following form

$$\begin{aligned} \textsf{data}\ \tau = \ & c_1 : \tau_{1,1} \times \ldots \times \tau_{1,m_1} \to \tau \\ |\ & \ldots \\ |\ & c_n : \tau_{n,1} \times \ldots \times \tau_{n,m_n} \to \tau \end{aligned}$$

  where
  - $\tau \notin \mathcal{T}y$ — fresh type name
  - $c_1, \ldots, c_n \notin \Sigma$ and $c_i \neq c_j$ for $i \neq j$ — fresh and distinct constructor names
  - each $\tau_{i,j} \in \{\tau\} \cup \mathcal{T}y$ — only known types
  - exists c_i such that $\tau_{i,j} \in \mathcal{T}y$ for all j — non-recursive constructor
- what could happen if one of the conditions is dropped?

Non-Empty Universes

- without the last condition (non-recursive constructor) the following data type declaration would be allowed (assuming that Nat and Succ are fresh names)

$$\mathsf{data}\ \mathsf{Nat} = \mathsf{Succ} : \mathsf{Nat} \to \mathsf{Nat}$$

  with the universe defined as the inductive set $\mathcal{A}_{\mathsf{Nat}}$

$$\frac{t \in \mathcal{A}_{\mathsf{Nat}}}{\mathsf{Succ}(t) \in \mathcal{A}_{\mathsf{Nat}}}$$

- consequence: $\mathcal{A}_{\mathsf{Nat}} = \varnothing$
- hence, non-recursive constructors are essential for having non-empty universes

Non-Empty Universes: Proof

Theorem

Let there be a list of data type declarations and an arbitrary type τ from this list. Then $\mathcal{A}_\tau \neq \varnothing$.

Proof

Let $\tau_1, \dots, \tau_n$ be the sequence of types that have been defined. We show

$$P(n) := \forall 1 \leq i \leq n.\ \mathcal{A}_{\tau_i} \neq \varnothing$$

by induction on n. This will entail the theorem.
In the base case we have to prove $P(0)$, which is trivially true. Now let us show $P(n+1)$ assuming $P(n)$. Because of $P(n)$, we only have to prove $\mathcal{A}_{\tau_{n+1}} \neq \varnothing$. By the definition of data types, there must be some $c_i : \tau_{i,1} \times \ldots \times \tau_{i,m_i} \to \tau_{n+1}$ where all $\tau_{i,j} \in \{\tau_1, \dots, \tau_n\}$. By the IH $P(n)$ we know that $\mathcal{A}_{\tau_{i,j}} \neq \varnothing$ for all j between 1 and m_i. Hence, there must be terms $t_1 \in \mathcal{A}_{\tau_{i,1}}, \dots, t_{m_i} \in \mathcal{A}_{\tau_{i,m_i}}$. Consequently, $c_i(t_1, \dots, t_{m_i}) \in \mathcal{A}_{\tau_{n+1}}$, and hence $\mathcal{A}_{\tau_{n+1}} \neq \varnothing$.

Current State

- presented: data type definitions
- semantics
  - free constructors: each constructor is interpreted as itself
  - universe as inductively defined sets: no infinite terms, such as infinite lists $\mathsf{Cons}(\mathsf{Zero}, \mathsf{Cons}(\mathsf{Zero}, \dots))$
    (modeling of infinite data structures would be possible via domain-theory)
- upcoming: functional programs, i.e., function definitions

Functional Programming – Function Definitions

Splitting the signature

- distinguish between
  - constructors, declared via data (start with capital letters in Haskell)
    e.g., Nil, Succ, Cons
  - defined functions, declared via equations (start with lowercase letters in Haskell)
    e.g., append, add, reverse
- formally, we have $\Sigma = \mathcal{C} \uplus \mathcal{D}$
- $\mathcal{C}$ is set of constructors, defined via data
  - constructors are written c, c_i, d in generic constructs such as data type definitions
  - start with uppercase letters in concrete examples (Succ, Cons)
- $\mathcal{D}$ is set of defined symbols, defined via function declarations
  - defined (function) symbols are written f, f_i, g in generic constructs such as function definitions
  - start with lowercase letters in concrete examples (append, reverse)
- we use F, G for elements of Σ whenever separation between $\mathcal{C}$ and $\mathcal{D}$ is not relevant
- note that in the standard model, $\mathcal{A}_\tau$ is exactly $\mathcal{T}(\mathcal{C})_\tau := \mathcal{T}(\mathcal{C}, \varnothing)_\tau$, which is the set of constructor ground terms of type τ

Notions for Preparing Function Definitions

- a pattern is a term in $\mathcal{T}(\mathcal{C}, \mathcal{V})$, usually written p or p_i
- a term t in $\mathcal{T}(\Sigma, \mathcal{V})$ is linear, if all variables within t occur only once
  - reverse(Cons(x, Cons(y, xs))) ✔
  - reverse(Cons(x, Cons(x, xs))) ✘
- the variables of a term t are defined as $\mathcal{V}ars(t)$
  - $\mathcal{V}ars(x) = \{x\}$
  - $\mathcal{V}ars(F(t_1, \ldots, t_n)) = \mathcal{V}ars(t_1) \cup \ldots \cup \mathcal{V}ars(t_n)$

Function Definitions

- besides data type definitions, a functional program consists of a sequence of function definitions, each having the following form

$$\begin{aligned} f &: \tau_1 \times \ldots \times \tau_n \to \tau \\ \ell_1 &= r_1 \\ \ldots &= \ldots \\ \ell_m &= r_m \end{aligned}$$

where

- f is a fresh name and $\mathcal{D} := \mathcal{D} \cup \{f : \tau_1 \times \ldots \times \tau_n \to \tau\}$ (hence, f is also added to $\Sigma = \mathcal{C} \cup \mathcal{D}$)
- each left-hand side (lhs) ℓ_i is linear
- each lhs ℓ_i is of the form $f(p_1, \ldots, p_n)$ with all p_j's being patterns
- each lhs ℓ_i and rhs r_i only use currently known symbols: $\ell_i, r_i \in \mathcal{T}(\Sigma, \mathcal{V})$
- each lhs ℓ_i and rhs r_i respect the type: $\ell_i, r_i \in \mathcal{T}(\Sigma, \mathcal{V})_\tau$
- each equation $\ell_i = r_i$ satisfies the variable condition $\mathcal{V}ars(r_i) \subseteq \mathcal{V}ars(\ell_i)$

Function Definitions: Examples

- assume data types Nat and List have been defined as before (slide 5)

$$\mathsf{add} : \mathsf{Nat} \times \mathsf{Nat} \to \mathsf{Nat}$$
$$\mathsf{add}(\mathsf{Zero}, y) = y$$
$$\mathsf{add}(\mathsf{Succ}(x), y) = \mathsf{add}(x, \mathsf{Succ}(y))$$

$$\mathsf{append} : \mathsf{List} \times \mathsf{List} \to \mathsf{List}$$
$$\mathsf{append}(\mathsf{Cons}(x, xs), ys) = \mathsf{Cons}(x, \mathsf{append}(xs, ys))$$
$$\mathsf{append}(xs, ys) = ys$$

$$\mathsf{head} : \mathsf{List} \to \mathsf{Nat}$$
$$\mathsf{head}(\mathsf{Cons}(x, xs)) = x$$

$$\mathsf{zeros} : \mathsf{List}$$
$$\mathsf{zeros} = \mathsf{Cons}(\mathsf{Zero}, \mathsf{zeros})$$

Function Definitions: Non-Examples

- assume program from previous slides + data Bool = True | False

$$\begin{aligned}
&\mathsf{even} : \mathsf{Nat} \to \mathsf{Bool} \\
&\mathsf{even}(\mathsf{Zero}) = \mathsf{True} \\
&\mathsf{even}(\mathsf{Succ}(x)) = \mathsf{odd}(x) && ✘ \\
&\mathsf{odd} : \mathsf{Nat} \to \mathsf{Bool} \\
&\mathsf{odd}(\mathsf{Zero}) = \mathsf{False} \\
&\mathsf{odd}(\mathsf{Succ}(x)) = \mathsf{even}(x) && ✘ \\
&\mathsf{random} : \mathsf{Nat} \\
&\mathsf{random} = x && ✘ \\
&\mathsf{minus} : \mathsf{Nat} \times \mathsf{Nat} \to \mathsf{Nat} \\
&\mathsf{minus}(\mathsf{Succ}(x), \mathsf{Succ}(y)) = \mathsf{minus}(x, y) \\
&\mathsf{minus}(x, \mathsf{Zero}) = x \\
&\mathsf{minus}(x, x) = \mathsf{Zero} && ✘ \\
&\mathsf{minus}(\mathsf{add}(x, y), x) = y && ✘
\end{aligned}$$

Semantics for Function Definitions

- problem: given a function definition

$$\begin{aligned} f &: \tau_1 \times \ldots \times \tau_n \to \tau \\ \ell_1 &= r_1 \\ \ldots &= \ldots \\ \ell_m &= r_m \end{aligned}$$

  we need to extend the semantics in the standard model, i.e., define the function

$$f^{\mathcal{M}} : \mathcal{A}_{\tau_1} \times \ldots \times \mathcal{A}_{\tau_n} \to \mathcal{A}_\tau$$

  or equivalently

$$f^{\mathcal{M}} : \mathcal{T}(\mathcal{C})_{\tau_1} \times \ldots \times \mathcal{T}(\mathcal{C})_{\tau_n} \to \mathcal{T}(\mathcal{C})_\tau$$

- idea: define $f^{\mathcal{M}}(t_1, \ldots, t_n)$ as

  the result of $f(t_1, \ldots, t_n)$ after evaluation w.r.t. equations in program

Semantics for Function Definitions – Continued

- required: $f^{\mathcal{M}} : \mathcal{T}(\mathcal{C})_{\tau_1} \times \ldots \times \mathcal{T}(\mathcal{C})_{\tau_n} \to \mathcal{T}(\mathcal{C})_{\tau}$
- idea: define $f^{\mathcal{M}}(t_1, \ldots, t_n)$ as

  the result of $f(t_1, \ldots, t_n)$ after evaluation w.r.t. equations in program
- several issues:
  - how is term evaluation defined?
    - briefly: replace instances of lhss by instances of rhss as long as possible
  - is result unique?
  - is result element of $\mathcal{T}(\mathcal{C})_{\tau}$?
  - does evaluation terminate?

Function Definitions: Examples

- consider previous program, type declarations omitted

$$\mathsf{add}(\mathsf{Zero}, y) = y \tag{1}$$
$$\mathsf{add}(\mathsf{Succ}(x), y) = \mathsf{add}(x, \mathsf{Succ}(y)) \tag{2}$$
$$\mathsf{append}(\mathsf{Cons}(x, xs), ys) = \mathsf{Cons}(x, \mathsf{append}(xs, ys)) \tag{3}$$
$$\mathsf{append}(xs, ys) = ys \tag{4}$$
$$\mathsf{head}(\mathsf{Cons}(x, xs)) = x \tag{5}$$
$$\mathsf{zeros} = \mathsf{Cons}(\mathsf{Zero}, \mathsf{zeros}) \tag{6}$$

- is result unique? no: consider $t = \mathsf{append}(\mathsf{Cons}(\mathsf{Zero}, \mathsf{Nil}), \mathsf{Nil})$

  then $t \overset{(3)}{=} \mathsf{Cons}(\mathsf{Zero}, \mathsf{append}(\mathsf{Nil}, \mathsf{Nil})) \overset{(4)}{=} \mathsf{Cons}(\mathsf{Zero}, \mathsf{Nil})$

  and $t \overset{(4)}{=} \mathsf{Nil}$
- is result element of $\mathcal{T}(\mathcal{C})_\tau$? no: $\mathsf{head}(\mathsf{Nil})$ cannot be evaluated
- does evaluation terminate? no: $\mathsf{zeros} = \mathsf{Cons}(\mathsf{Zero}, \mathsf{zeros}) = \dots$
- solution: further restrictions on function definitions

Functional Programming – Operational Semantics

Functional Programming: Operational Semantics

- operational semantics: formal definition on how evaluation proceeds step-by-step
- main operation: applying a substitution $\sigma : \mathcal{V} \to \mathcal{T}(\Sigma, \mathcal{V})$ to a term, can be defined recursively
  - $x\sigma = \sigma(x)$
  - $F(t_1, \dots, t_n)\sigma = F(t_1\sigma, \dots, t_n\sigma)$
- one-step evaluation relation $\hookrightarrow \subseteq \mathcal{T}(\Sigma, \mathcal{V}) \times \mathcal{T}(\Sigma, \mathcal{V})$ defined as inductive set

$$\frac{\ell = r \text{ is equation in program}}{\ell\sigma \hookrightarrow r\sigma} \text{ root step}$$

$$\frac{F \in \Sigma \quad s_i \hookrightarrow t_i}{F(s_1, \dots, s_i, \dots, s_n) \hookrightarrow F(s_1, \dots, t_i, \dots, s_n)} \text{ rewrite in context}$$

- given a term t and a lhs ℓ, for checking whether a root-step is applicable one needs matching: $\exists \sigma.\, \ell\sigma = t$ (and also deliver that σ)
- same evaluation as in functional programming (lecture), except that order of equations is ignored and here it becomes formal

Matching

- we define matching as an operation on a set of pairs $P = \{(\ell_1, t_1), \dots, (\ell_n, t_n)\}$ and the task is to decide: $\exists \sigma.\, \ell_1\sigma = t_1 \wedge \dots \wedge \ell_n\sigma = t_n$, i.e.,
  - either return the required substitution σ in the form of a set of pairs $\{(x_1, s_1), \dots, (x_m, s_m)\}$ with all x_i distinct which can then be interpreted as the substitution σ defined by

$$\sigma(x) = \begin{cases} s_i, & \text{if } x = x_i \text{ for some } i \\ x, & \text{otherwise} \end{cases}$$

  - or return $\bot$ indicating that no such substitution exists
- matching algorithm: apply rules $\curvearrowright$ as long as possible

$$P \uplus \{(F(\ell_1, \dots, \ell_n), F(t_1, \dots, t_n))\} \curvearrowright P \cup \{(\ell_1, t_1), \dots, (\ell_n, t_n)\} \qquad \text{(decompose)}$$

$$P \uplus \{(F(\dots), G(\dots))\} \curvearrowright \bot \qquad \text{if } F \neq G \qquad \text{(clash)}$$

$$P \uplus \{(F(\dots), x)\} \curvearrowright \bot \qquad \text{if } x \in \mathcal{V} \qquad \text{(fun-var)}$$

$$P \uplus \{(x, s), (x, t)\} \curvearrowright \bot \qquad \text{if } x \in \mathcal{V} \text{ and } s \neq t \qquad \text{(var-clash)}$$

Matching – Example

- we want to test whether there is a root step possible for the term $t = \mathsf{append}(\mathsf{Cons}(y, \mathsf{Nil}), \mathsf{Cons}(y, ys))$ w.r.t. the equation $(\ell = r) = (\mathsf{append}(\mathsf{Cons}(x, xs), ys) = \mathsf{Cons}(x, \mathsf{append}(xs, ys)))$
- setup matching problem $\{(\ell, t)\}$

$$\begin{aligned} P = {} & \{(\mathsf{append}(\mathsf{Cons}(x, xs), ys), \mathsf{append}(\mathsf{Cons}(y, \mathsf{Nil}), \mathsf{Cons}(y, ys)))\} \\ \curvearrowright {} & \{(\mathsf{Cons}(x, xs), \mathsf{Cons}(y, \mathsf{Nil})), (ys, \mathsf{Cons}(y, ys))\} \\ \curvearrowright {} & \{(x, y), (xs, \mathsf{Nil}), (ys, \mathsf{Cons}(y, ys))\} \end{aligned}$$

- obtain substitution $\sigma(z) = \begin{cases} y, & \text{if } z = x \\ \mathsf{Nil}, & \text{if } z = xs \\ \mathsf{Cons}(y, ys), & \text{if } z = ys \\ z, & \text{otherwise} \end{cases}$
- so, $t = \ell\sigma \hookrightarrow r\sigma = \mathsf{Cons}(x, \mathsf{append}(xs, ys))\sigma = \mathsf{Cons}(y, \mathsf{append}(\mathsf{Nil}, \mathsf{Cons}(y, ys)))$

Matching – Verification and Termination Proof

- matching algorithm

$$P \uplus \{(F(\ell_1,\dots,\ell_n), F(t_1,\dots,t_n))\} \curvearrowright P \cup \{(\ell_1,t_1),\dots,(\ell_n,t_n)\} \qquad \text{(decompose)}$$

$$P \uplus \dots \curvearrowright \bot \qquad \text{(other rules)}$$

- soundness = termination + partial correctness
- termination: in each step, the sum of the size of terms (# symbols) is decreased

$$\begin{aligned}
|(F(\ell_1,\dots,\ell_n), F(t_1,\dots,t_n))| &= |F(\ell_1,\dots,\ell_n)| + |F(t_1,\dots,t_n)| \\
&= 1 + \sum_i |\ell_i| + 1 + \sum_i |t_i| \\
&> \sum_i |\ell_i| + \sum_i |t_i| \\
&= \sum_i |(\ell_i, t_i)|
\end{aligned}$$

Matching – Type Preservation

- matching algorithm

$$P \uplus \{(F(\ell_1,\dots,\ell_n), F(t_1,\dots,t_n))\} \curvearrowright P \cup \{(\ell_1,t_1),\dots,(\ell_n,t_n)\} \qquad \text{(decompose)}$$
$$P \uplus \dots \curvearrowright \bot \qquad \text{(other rules)}$$

- property: we say that a set of pairs P is type-correct, iff for all pairs $(\ell, t) \in P$ the types of ℓ and t are identical, i.e., $\exists \tau. \{\ell, t\} \subseteq \mathcal{T}(\Sigma, \mathcal{V})_\tau$
- theorem: whenever P is type-correct, then P will stay type-correct during the algorithm; consequently, any result $\neq \bot$ will be type-correct
- proof: we prove an invariant, so we only need to prove that the property is maintained when performing a single $\curvearrowright$-step in the algorithm:
  consider "decompose"
  - we can assume $\{F(\ell_1,\dots,\ell_n), F(t_1,\dots,t_n)\} \subseteq \mathcal{T}(\Sigma, \mathcal{V})_\tau$
  - so $F : \tau_1 \times \ldots \times \tau_n \to \tau$ for suitable τ_i
  - hence, $\{\ell_i, t_i\} \subseteq \mathcal{T}(\Sigma, \mathcal{V})_{\tau_i}$ for all i

Matching – Structure of Result

- matching algorithm: apply $\curvearrowright$ as long as possible

$$P \uplus \{(F(\ell_1,\dots,\ell_n),F(t_1,\dots,t_n))\} \curvearrowright P \cup \{(\ell_1,t_1),\dots,(\ell_n,t_n)\} \qquad \text{(decompose)}$$

$$P \uplus \{(F(\dots),G(\dots))\} \curvearrowright \bot \qquad \text{if } F \neq G \qquad \text{(clash)}$$

$$P \uplus \{(F(\dots),x)\} \curvearrowright \bot \qquad \text{if } x \in \mathcal{V} \qquad \text{(fun-var)}$$

$$P \uplus \{(x,s),(x,t)\} \curvearrowright \bot \qquad \text{if } x \in \mathcal{V} \text{ and } s \neq t \qquad \text{(var-clash)}$$

- property: result of matching algorithm on well-typed inputs is $\bot$ or set $\{(x_1,s_1),\dots,(x_m,s_m)\}$ with all x_i distinct
- proof
  - assume result is not $\bot$, then it must be some set of pairs $P = \{(u_1,s_1),\dots,(u_m,s_m)\}$ where no rule is applicable
  - if all u_i's are variables, then the result follows: there cannot be two entries (u_i,s_i) and (u_j,s_j) with $u_i = u_j$ and $s_i \neq s_j$ because then "var-clash" would have been applied
  - it remains to consider the case that some $u_i = F(\ell_1,\dots,\ell_n)$
  - $s_i = F(t_1,\dots,t_k)$, as result is not $\bot$, cf. "clash" and "fun-var"
  - then $k = n$ because of type preservation: contradiction to "decompose"

Matching – Preservation of Solutions

- matching algorithm

$$P \uplus \{(F(\ell_1,\dots,\ell_n), F(t_1,\dots,t_n))\} \curvearrowright P \cup \{(\ell_1,t_1),\dots,(\ell_n,t_n)\} \qquad \text{(decompose)}$$

$$P \uplus \{(F(\ldots), G(\ldots))\} \curvearrowright \bot \qquad \text{if } F \neq G \qquad \text{(clash)}$$

$$P \uplus \{(F(\ldots), x)\} \curvearrowright \bot \qquad \text{if } x \in \mathcal{V} \qquad \text{(fun-var)}$$

$$P \uplus \{(x,s),(x,t)\} \curvearrowright \bot \qquad \text{if } x \in \mathcal{V} \text{ and } s \neq t \qquad \text{(var-clash)}$$

- property: algorithm preserves matching substitutions (where $\bot$ has no matching substitution)
- proof by considering invariant of single step: whenever $P \curvearrowright P'$, then σ is a matcher of P iff σ is matcher of P'
  - clash: both "σ is matcher of $\{(F(\ldots),G(\ldots))\} \cup P$" and "$\sigma$ is matcher of $\bot$" are wrong: $F(t_1,\dots)\sigma = F(t_1\sigma,\dots) \neq G(\ldots)$
  - fun-var and var-clash are similar
  - decompose: $F(\ell_1,\dots,\ell_n)\sigma = F(t_1,\dots,t_n)$
    $\longleftrightarrow F(\ell_1\sigma,\dots,\ell_n\sigma) = F(t_1,\dots,t_n)$
    $\longleftrightarrow \ell_1\sigma = t_1 \wedge \ldots \wedge \ell_n\sigma = t_n$

Matching Algorithm – Summary

- algorithm: apply $\curvearrowright$ as long as possible
- (one) termination proof
- (many) partial correctness proofs
  mainly by showing invariants that are preserved by $\curvearrowright$
  - type preservation
  - preservation of matching substitutions
  - result is $\bot$ or a set which encodes a substitution
- application: compute root steps by testing whether decomposition of term into $\ell\sigma$ for equation $\ell = r$ is possible
- core of functional programming (and term rewriting)
- much better algorithms exists, which avoid to match against all lhss, based on precalculation (term indexing), e.g., group equations by root symbol of lhss

Semantics in the Standard Model

Towards Semantics in Standard Model

- evaluation of terms is now explained: one-step relation $\hookrightarrow$
- algorithm for evaluation is similar to matching algorithm:

  apply $\hookrightarrow$-steps until no longer possible
- questions are similar as in matching algorithm
  - termination: do we always get result?
  - preservation of types?
  - is result a desired value, i.e., a constructor ground term?
  - is result unique?
- questions don't have positive answer in general, cf. slide 20

Type Preservation of $\hookrightarrow$

- aim: show that $\hookrightarrow$ preserves types:

$$t \in \mathcal{T}(\Sigma, \mathcal{V})_\tau \longrightarrow t \hookrightarrow s \longrightarrow s \in \mathcal{T}(\Sigma, \mathcal{V})_\tau$$

- proof will be by induction w.r.t. inductively defined set $\hookrightarrow$ for arbitrary τ
- preliminary: we call a substitution type-correct, if $\sigma(x) \in \mathcal{T}(\Sigma, \mathcal{V})_\tau$ whenever $x : \tau \in \mathcal{V}$
- easy result: whenever $t \in \mathcal{T}(\Sigma, \mathcal{V})_\tau$ and σ is type-correct, then $t\sigma \in \mathcal{T}(\Sigma, \mathcal{V})_\tau$
(how would you prove it?)

Type Preservation of $\hookrightarrow$ – Proof

- proof: induction w.r.t. inductively defined set $\hookrightarrow$ for arbitrary τ
- base case: $\ell\sigma \hookrightarrow r\sigma$ for some equation $\ell = r$ of the program where $\ell\sigma \in \mathcal{T}(\Sigma, \mathcal{V})_\tau$ and we have to prove $r\sigma \in \mathcal{T}(\Sigma, \mathcal{V})_\tau$
  - since $\ell\sigma \in \mathcal{T}(\Sigma, \mathcal{V})_\tau$, and $\ell, r \in \mathcal{T}(\Sigma, \mathcal{V})_\tau$ by the definition of functional programs, we conclude that σ is type-correct, cf. slide 26
  - and since $r \in \mathcal{T}(\Sigma, \mathcal{V})_\tau$ and σ is type-correct, then also $r\sigma \in \mathcal{T}(\Sigma, \mathcal{V})_\tau$, cf. previous slide
- step case: $F(s_1, \ldots, s_i, \ldots, s_n) \hookrightarrow F(s_1, \ldots, t_i, \ldots, s_n)$ since $s_i \hookrightarrow t_i$, we know $F(s_1, \ldots, s_i, \ldots, s_n) \in \mathcal{T}(\Sigma, \mathcal{V})_\tau$ and have to prove $F(s_1, \ldots, t_i, \ldots, s_n) \in \mathcal{T}(\Sigma, \mathcal{V})_\tau$
  - since $F(s_1, \ldots, s_i, \ldots, s_n) \in \mathcal{T}(\Sigma, \mathcal{V})_\tau$, we know that $F : \tau_1 \times \ldots \times \tau_n \to \tau \in \Sigma$ and each $s_j \in \mathcal{T}(\Sigma, \mathcal{V})_{\tau_j}$ for $1 \leq j \leq n$
  - by the IH we know $t_i \in \mathcal{T}(\Sigma, \mathcal{V})_{\tau_i}$ – note that here we can take a different type than τ, namely τ_i, because the induction was for arbitrary τ
  - but then we immediately conclude $F(s_1, \ldots, t_i, \ldots, s_n) \in \mathcal{T}(\Sigma, \mathcal{V})_\tau$

Type Preservation of $\hookrightarrow^*$

- finally, we can show that evaluation (execution of arbitrarily many $\hookrightarrow$-steps, written $\hookrightarrow^*$) preserves types, which is an easy induction proof on the number of steps by using type-preservation of $\hookrightarrow$
- theorem: whenever $t \in \mathcal{T}(\Sigma, \mathcal{V})_\tau$ and $t \hookrightarrow^* s$, then $s \in \mathcal{T}(\Sigma, \mathcal{V})_\tau$
- proofs to obtain global result
  1. show that matching preserves types (slide 26)
     proof via invariant, since matching algorithm is imperative (while rules-applicable ...)
  2. show that substitution application preserves types (slide 31)
     proof by induction on terms, following recursive structure of definition of substitution application (slide 22)
  3. show that $\hookrightarrow$ preserves types (slide 33)
     proof by structural induction w.r.t. inductively defined set $\hookrightarrow$;
     uses results 1 and 2
  4. show that $\hookrightarrow^*$ preserves types
     proof on number of steps; uses result 3

Preservation of Groundness of $\hookrightarrow^*$

- a term t is ground if $\mathcal{V}ars(t) = \varnothing$, or equivalently if $t \in \mathcal{T}(\Sigma)$
- recall aim: we want to evaluate ground term like $\mathsf{append}(\mathsf{Cons}(\mathsf{Zero}, \mathsf{Nil}), \mathsf{Nil})$ to element of universe, i.e., constructor ground term
- hence, we need to ensure that result of evaluation with $\hookrightarrow$ is ground
- preservation of groundness can be shown with similar proof structure as in the proof of preservation of types

Normal Forms – The Results of an Evaluation

- a term t is a normal form (w.r.t. $\hookrightarrow$) if no further $\hookrightarrow$-steps are possible:

$$\nexists s.\ t \hookrightarrow s$$

- whenever $t \hookrightarrow^* s$ and s is in normal form, then we write

$$t \hookrightarrow^! s$$

  and call s a normal form of t
- normal forms represent the result of an evaluation
- known results at this point: whenever $t \in \mathcal{T}(\Sigma)_\tau$ and $t \hookrightarrow^! s$ then
  - $s \in \mathcal{T}(\Sigma, \mathcal{V})_\tau$ (type-preservation)
  - $s \in \mathcal{T}(\Sigma)$ (groundness-preservation)
  - $s \in \mathcal{T}(\Sigma)_\tau$ (combined)
- missing:
  - $s \in \mathcal{T}(\mathcal{C})_\tau$ (constructor-ground term)
  - s is unique
  - s always exists

Pattern Completeness

- a function symbol $f : \tau_1 \times \ldots \times \tau_n \to \tau \in \mathcal{D}$ is pattern complete iff for all $t_1 \in \mathcal{T}(\mathcal{C})_{\tau_1}$, …, $t_n \in \mathcal{T}(\mathcal{C})_{\tau_n}$ there is an equation $\ell = r$ in the program, such that ℓ matches $f(t_1, \ldots, t_n)$
- a functional program is pattern complete iff all $f \in \mathcal{D}$ are pattern complete
- example

$$\mathsf{append}(\mathsf{Cons}(x, xs), ys) = \mathsf{Cons}(x, \mathsf{append}(xs, ys))$$
$$\mathsf{append}(\mathsf{Nil}, ys) = ys$$
$$\mathsf{head}(\mathsf{Cons}(x, xs)) = x$$

  - append is pattern complete
  - head is not pattern complete: for head(Nil) there is no matching lhs

Pattern Completeness and Constructor Ground Terms

- theorem: if a program is pattern complete and $t \in \mathcal{T}(\Sigma)_\tau$ is a normal form, then $t \in \mathcal{T}(\mathcal{C})_\tau$
- proof of $P(t, \tau)$ by structural induction w.r.t. $\mathcal{T}(\Sigma)_\tau$ for

$$P(t, \tau) := t \text{ is normal form} \longrightarrow t \in \mathcal{T}(\mathcal{C})_\tau$$

  - induction yields only one case: $t = F(t_1, \ldots, t_n)$ where $F : \tau_1 \times \ldots \times \tau_n \to \tau \in \Sigma$
  - IH for each i: if t_i is normal form, then $t_i \in \mathcal{T}(\mathcal{C})_{\tau_i}$
  - premise: $F(t_1, \ldots, t_n)$ is normal form
  - from premise conclude that t_i is normal form: (if $t_i \hookrightarrow s_i$ then $F(t_1, \ldots, t_n) \hookrightarrow F(t_1, \ldots, s_i, \ldots, t_n)$ shows that $F(t_1, \ldots, t_n)$ is not a normal form)
  - in combination with IH: each $t_i \in \mathcal{T}(\mathcal{C})_{\tau_i}$
  - consider two cases: $F \in \mathcal{C}$ or $F \in \mathcal{D}$
  - case $F \in \mathcal{C}$: using $t_i \in \mathcal{T}(\mathcal{C})_{\tau_i}$ immediately yields $F(t_1, \ldots, t_n) \in \mathcal{T}(\mathcal{C})_\tau$
  - case $F \in \mathcal{D}$: using pattern completeness and $t_i \in \mathcal{T}(\mathcal{C})_{\tau_i}$, conclude that $F(t_1, \ldots, t_n)$ must be matched by lhs; this is contradiction to $F(t_1, \ldots, t_n)$ being a normal form

Pattern Disjointness

- a function symbol $f : \tau_1 \times \ldots \times \tau_n \to \tau \in \mathcal{D}$ is pattern disjoint iff for all $t_1 \in \mathcal{T}(\mathcal{C})_{\tau_1}$, $\ldots$, $t_n \in \mathcal{T}(\mathcal{C})_{\tau_n}$ there is at most one equation $\ell = r$ in the program, such that ℓ matches $f(t_1, \ldots, t_n)$
- a functional program is pattern disjoint iff all $f \in \mathcal{D}$ are pattern disjoint
- example

$$\mathsf{append}(\mathsf{Cons}(x, xs), ys) = \mathsf{Cons}(x, \mathsf{append}(xs, ys))$$
$$\mathsf{append}(xs, ys) = ys$$
$$\mathsf{head}(\mathsf{Cons}(x, xs)) = x$$

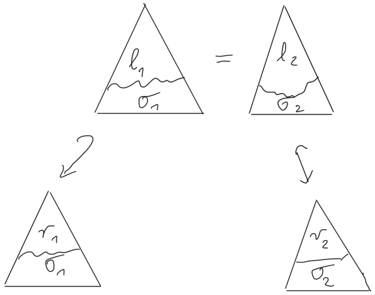
  - head is pattern disjoint
  - append is not pattern disjoint: the term $\mathsf{append}(\mathsf{Cons}(\mathsf{Zero}, \mathsf{Nil}), \mathsf{Nil})$ is matched by the lhss of both append-equations

Pattern Disjointness and Unique Normal Forms

- theorem: if a program is pattern disjoint then $\hookrightarrow$ is confluent and each term has at most one normal form
- confluence: whenever $s \hookrightarrow^* t$ and $s \hookrightarrow^* u$ then there exists some v such that $t \hookrightarrow^* v$ and $u \hookrightarrow^* v$
- proof of theorem:
  - pattern disjointness in combination with the other syntactic restrictions on functional programs implies that the defining equations form an orthogonal term rewrite sytem
  - Rosen proved that orthogonal term rewrite sytems are confluent
  - confluence implies that each term has at most one normal form
  - full proof of Rosen given in term rewriting lecture, we only sketch a weaker property on the next slides, namely local confluence: whenever $s \hookrightarrow t$ and $s \hookrightarrow u$ then there exists some v such that $t \hookrightarrow^* v$ and $u \hookrightarrow^* v$
  - local confluence in combination with termination also implies confluence

Proof of Local Confluence: Two Root Steps

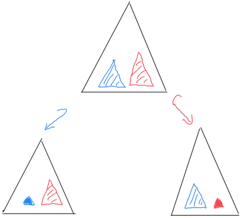
- consider the situation in the diagram where two root steps with equations $\ell_1 = r_1$ and $\ell_2 = r_2$ are applied



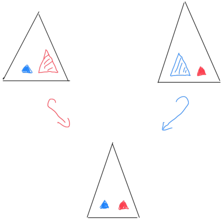
- because of pattern disjointness: $(\ell_1 = r_1) = (\ell_2 = r_2)$
- uniqueness of matching: $\sigma_1(x) = \sigma_2(x)$ for all $x \in \mathcal{V}ars(\ell_{1/2})$
- variable condition of programs: $\sigma_1(x) = \sigma_2(x)$ for all $x \in \mathcal{V}ars(r_{1/2})$
- hence $r_1\sigma_1 = r_2\sigma_2$

Proof of Local Confluence: Independent Steps

- consider the situation in the diagram where two steps at independent positions are applied

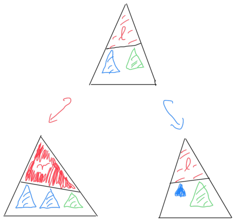


- just do the steps in reverse order

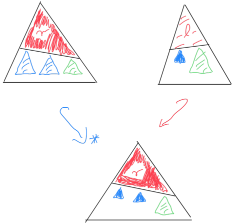


Proof of Local Confluence: Root- and Substitution-Step

- consider the situation in the diagram where a root step overlaps with a step done in the substitution



- just do the steps in reverse order (perhaps multiple times)



Graphical Local Confluence Proof

- the diagrams in the three previous slides describe all situations where one term can be evaluated in two different ways (within one step)
- in all cases the diagrams could be joined
- overall: intuitive graphical proof of local confluence
- often hard task: transform such an intuitive proof into a formal, purely textual proof, using induction, case-analysis, etc.

Semantics for Functional Programs in the Standard Model

- we are now ready to complete the semantics for functional programs
- we call a functional program well-defined, if
  - it is pattern disjoint,
  - it is pattern complete, and
  - $\hookrightarrow$ is terminating
- for well-defined programs, we define for each $f : \tau_1 \times \ldots \times \tau_n \to \tau \in \mathcal{D}$

$$
\begin{aligned}
&f^{\mathcal{M}} : \mathcal{T}(\mathcal{C})_{\tau_1} \times \ldots \times \mathcal{T}(\mathcal{C})_{\tau_n} \to \mathcal{T}(\mathcal{C})_{\tau} \\
&f^{\mathcal{M}}(t_1, \ldots, t_n) = s
\end{aligned}
$$

  where s is the unique normal form of $f(t_1, \ldots, t_n)$, i.e., $f(t_1, \ldots, t_n) \hookrightarrow^! s$
- remarks:
  - a normal form exists, since $\hookrightarrow$ is terminating
  - s is unique because of pattern disjointness
  - $s \in \mathcal{T}(\mathcal{C})_\tau$ because of pattern completeness, and type- and groundness-preservation

Summary: Standard Model

- standard model
  - universes: $\mathcal{T}(\mathcal{C})_\tau$
  - constructors: $c^{\mathcal{M}}(t_1, \ldots, t_n) = c(t_1, \ldots, t_n)$
  - defined symbols: $f^{\mathcal{M}}(t_1, \ldots, t_n)$ is normal form of $f(t_1, \ldots, t_n)$ w.r.t. $\hookrightarrow$
- if functional program is well-defined
  - pattern disjoint,
  - pattern complete, and
  - $\hookrightarrow$ is terminating

  then standard model is well-defined
- upcoming
  - what about functional programs that are not well-defined?
  - comparison to real functional programming languages
  - treatment in real proof assistants

Without Pattern Disjointness

- consider Haskell program

```
conj :: Bool -> Bool -> Bool
conj True True = True     -- (1)
conj x    y    = False    -- (2)
```

- obviously not pattern disjoint
- however, Haskell still has unique results, since equations are ordered
  - an equation is only applicable
    if all previous equations are not applicable
  - so, `conj True True` can only be evaluated to `True`
- ordering of equations can be resolved by instantiation equations via complementary patterns
- equivalent equations (in Haskell) which do not rely upon order of equations

```
conj :: Bool -> Bool -> Bool
conj True  True  = True   -- (1)
conj False y     = False  -- (2) with x / False
conj True  False = False  -- (2) with x / True, y / False
```

Without Pattern Disjointness – Continued

- pattern disjointness is sufficient criterion to ensure confluence
- overlaps can be allowed, if they do not cause conflicts
- example:

  ```
  conj :: Bool -> Bool -> Bool
  conj True  True  = True
  conj False y     = False   -- (1)
  conj x     False = False   -- (2)
  ```

  the only overlap is `conj False False`; it is harmless since the term evaluates to the same result using both (1) and (2)
- translating ordered equations into pattern disjoint equations or equations which only have harmless overlaps can be done automatically
  - usually, there are several possibilities
  - finding the smallest set of equations is hard
  - automatically done in proof-assistants such as Isabelle;
    e.g., overlapping `conj` from previous slide is translated into above one
- consequence: pattern disjointness is no real restriction

Without Pattern Completeness

- pattern completeness is naturally missing in several functions
- examples from Haskell libraries

```
head :: [a] -> a
head (x : xs) = x
```

- resolving pattern incompleteness is possible in the standard model
  - determine missing patterns
  - add for these missing cases equations that assign some element of the universe

$$\mathsf{head}(\mathsf{Cons}(x, xs)) = x \qquad \text{equation as before}$$

$$\mathsf{head}(\mathsf{Nil}) = \text{some element of } \mathcal{T}(\mathcal{C})_{\mathsf{Nat}} \qquad \text{new equation}$$

  - in this way, head becomes pattern complete and $\mathsf{head}^{\mathcal{M}}$ is total
  - "some element" really is an element of $\mathcal{T}(\mathcal{C})_{\mathsf{Nat}}$, and not a special error value like $\bot$
  - the added equation with "some element" is usually not revealed to the user, so the user cannot infer what number $\mathsf{head}(\mathsf{Nil})$ actually is
- consequence: pattern completeness is no real restriction

Without Termination

- definition of standard model just doesn't work properly in case of non-termination
- one possibility: use Scott's domain theory where among others, explicit $\bot$-elements are added to universe
- examples
  - $\mathcal{A}_{\mathsf{Nat}} = \{\bot, \mathsf{Zero}, \mathsf{Succ}(\mathsf{Zero}), \mathsf{Succ}(\mathsf{Succ}(\mathsf{Zero})), \dots, \mathsf{Succ}^{\infty}\}$
  - $\mathcal{A}_{\mathsf{List}} = \{\bot, \mathsf{Nil}, \mathsf{Cons}(\mathsf{Zero}, \mathsf{Nil}), \mathsf{Cons}(\bot, \mathsf{Nil}), \mathsf{Cons}(\bot, \bot), \dots\}$
- then semantics can be given to non-terminating computations
  - $\mathsf{inf} = \mathsf{Succ}(\mathsf{inf})$ leads to $\mathsf{inf}^{\mathcal{M}} = \mathsf{Succ}^{\infty}$
  - $\mathsf{undef} = \mathsf{undef}$ leads to $\mathsf{undef}^{\mathcal{M}} = \bot$
- problem: certain equalities don't hold w.r.t. domain theory semantics
  - assume usual definition of program for minus, then $\forall x.\, \mathsf{minus}(x, x) = \mathsf{Zero}$ is not true, consider $x = \mathsf{inf}$ or $x = \mathsf{undef}$
- since reasoning in domain theory is more complex, in this course we restrict to terminating functional programs
- even large proof assistants like Isabelle and Coq usually restrict to terminating functions for that reason

Summary of Part 3

- definition of well-defined functional programs
  - datatypes and function definitions (first order)
  - type-preserving equations within simple type system
  - well-defined: terminating, pattern complete and pattern disjoint
- definition of operational semantics $\hookrightarrow$
- definition of standard model
- upcoming
  - part 4: detect well-definedness, in particular termination
  - part 5: inference rules for standard model, equational reasoning engine