This is a reformatted version of an exam in WS 2022/2023. In total you can obtain 90 points and you would need at least 45 points to pass.

1 Algorithms in Linear Arithmetic

Consider the following formula:

$$\varphi := \exists x.\exists y.\ (3x + 2y < 2 \wedge -x + 5y > 1)$$

[10] (a) Remove the quantifier of y within φ using Cooper's method. You *don't* have to simplify the formula after the removal of the quantifier.

[10] (b) Start to solve φ using the simplex method: apply all initial steps and *one* iteration of the main loop. Use Bland's selection rule with the variable order $x < y < s < t$ where s and t are the introduced slack variables.
Is a second iteration of the main loop required? Just answer this question with a yes or no.

2 Understanding Linear Arithmetic

For some optimization problems it is required to consider *mixed* linear arithmetic problems, where the set of variables $\mathcal{V}$ is partitioned into $\mathcal{V}_\mathbb{Z} \uplus \mathcal{V}_\mathbb{Q}$, i.e., some variables are restricted to represent integers whereas the others are for rational numbers. For instance $\mathcal{V}_\mathbb{Z}$ might describe quantities which cannot be split, such as wheels, screws, etc., whereas $\mathcal{V}_\mathbb{Q}$ is used for amounts of fluids, for costs, etc.

A mixed solution of a quantifier-free formula φ is an assignment $v : \mathcal{V} \to \mathbb{Q}$ that satisfies φ and where additionally $v(x) \in \mathbb{Z}$ for all $x \in \mathcal{V}_\mathbb{Z}$.

[8] (a) Modify the branch-and-bound algorithm for LIA so that it can treat mixed linear arithmetic constraints, i.e., write down pseudo-code.

(b) Consider the small model property of LIA.

The existing proof translates a set of constraints with n variables given as a polyhedron $\{\vec{x} \mid A\vec{x} \leq \vec{b}\}$ with $A \in \mathbb{Z}^{m\times n}$ and $\vec{b} \in \mathbb{Z}^m$ into $\mathit{hull}(H) + \mathit{cone}(C)$ for some $H \subseteq \mathbb{Q}^n$ and $C \subseteq \mathbb{Z}^n$ where additionally some upper bound u is constructed in a way that all coefficients c of all vectors in $H \cup C$ are bounded: $|c| \leq u$.

Afterwards it is shown how some arbitrary integral solution $\vec{x} \in \mathit{hull}(H) + \mathit{cone}(C)$ can be turned into a small integral solution $\vec{y} \in \mathit{hull}(H) + \mathit{cone}(C)$ where $|y_i|$ is bounded by some expression involving u and n for all coefficients y_i of $\vec{y}$.

[4] i. Provide an upper bound for $|y_i|$ depending on u and n.

[8] ii. Does the transformation of the arbitrary integer solution $\vec{x}$ into the small integer solution $\vec{y}$ also work correctly in the mixed case with $\mathcal{V}_\mathbb{Z} = \{x_1, \dots, x_k\}$ and $\mathcal{V}_\mathbb{Q} = \{x_{k+1}, \dots, x_n\}$, i.e., when replacing the conditions of $\vec{x}, \vec{y}$ being integral by $\vec{x}, \vec{y} \in \mathbb{Z}^k \times \mathbb{Q}^{n-k}$? Explain your answer.

3 Algorithms for SAT

Consider the following clauses.

(a) $1 \vee \neg 7 \vee 8$

(b) $\neg 1 \vee \neg 2$

(c) $2 \vee 3$

(d) $\neg 1 \vee 4$

(e) $2 \vee \neg 4 \vee 5$

(f) $6 \vee \neg 9 \vee 10$

(g) $\neg 4 \vee \neg 6 \vee \neg 7$

(h) $\neg 6 \vee 8$

(i) $\neg 5 \vee 7 \vee \neg 8 \vee \neg 9$

(j) $\neg 5 \vee 9 \vee \neg 10$

(k) $9 \vee 11$

(l) $10 \vee \neg 11$

Further consider the following run of DPLL

$$\overset{d}{1}\ \underset{(b)}{\neg 2}\ \underset{(c)}{3}\ \underset{(d)}{4}\ \underset{(e)}{5}\ \overset{d}{6}\ \underset{(g)}{\neg 7}\ \underset{(h)}{8}\ \underset{(i)}{\neg 9}\ \underset{(j)}{\neg 10}\ \underset{(k)}{11}$$

where in this configuration a conflict w.r.t. clause (l) is detected.

[12] (a) Compute the implication graph from the beginning up to the conflict detection. Layout hint: put nodes 1, ¬2, 3 in one line at the top, place 4 below 1 and 6 below 4.

[4] (b) Identify the first unique implication point and write down the corresponding backjump clause.

[4] (c) Write down the next configuration that is obtain from applying the backjump rule w.r.t. the identified backjump clause.

4 Encoding Problems

Consider a puzzle game where each puzzle consists of several puzzle constraints p_i which are all of the form

$$c = 1 \cdot x_1 + 2 \cdot x_2 + \ldots + k \cdot x_k$$

where $c \in \mathbb{N}$ and $x_1, \ldots, x_k$ are variables chosen from a larger set of variables. A solution to a puzzle constraint must satisfy the equation and additionally the restriction that each variable x_i gets assigned a digit in the range from 0 to 7.

For instance, the constraint $14 = x + 2y + 3z$ can be solved by choosing $x = 7, y = 2, z = 1$ or $x = 0, y = 7, z = 0$ or …, but neither $x = 8, y = 3, z = 0$ nor $x = 1, y = 2, z = 4$.

Example: given two puzzle constraints $2 = 1x + 2y$ and $25 = 1y + 2x + 3z$ using variables x, y, z, this game has the unique solution $x = 2, y = 0, z = 7$.

[4] (a) Choose a theory (such as equality logic, difference logic, EUF, LRA, LIA, BV) and encode a single puzzle constraint as a formula which is as succinct as possible.

[4] (b) Choose another theory and encode a single puzzle constraint. You can reuse earlier formulas if desired.

[4] (c) Choose yet another theory and encode a single puzzle constraint.

[8] (d) Assume you are given a full puzzle game with puzzle constraints $p_1, \ldots, p_n$ using variables $x_1, \ldots, x_m$. Describe an SMT-based algorithm to check whether this game has a unique solution.

5 Multiple Choice

[10] Each correct answer is worth 2 points. Each wrong answer is worth -1 point. Giving no answer to a question is worth 0 points. A total negative score is pruned to 0 points.

Question	Yes	No
The Nelson–Oppen algorithm is a quantifier elimination algorithm for LRA.	☐	☐
The congruence closure algorithm is used in the context of EUF.	☐	☐
Difference logic constraints can be solved in polynomial time for both $\mathbb{Z}$ and $\mathbb{Q}$.	☐	☐
All of LRA, LIA and EUF are convex theories.	☐	☐
Array Logic is decidable if both the index theory (including quantifiers) is decidable and the element theory is decidable.	☐	☐