1 (a) *answer + explanation*

i. The function f is not monotone because $f(1,0) = 1$ and $f(1,1) = 0$. So $f(1,x) = \overline{x}$.

ii. The function f is not self-dual because $f(0,0) = 0 = f(1,1)$.

iii. We compute the algebraic normal form of f:

$$\begin{aligned} f(x,y) &= (x+y) \oplus y \\ &= (x \oplus y \oplus xy) \oplus y \\ &= x \oplus xy \end{aligned}$$

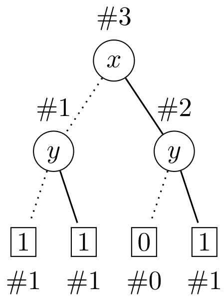
Since $x \oplus xy$ is not linear, f is not affine.

(b) *answer + explanation*

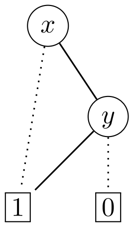
From the table

x	y	$f(x,y)$	$g(x,y)$	$\overline{f(x,y)} + g(x,y)$
0	0	0	0	1
0	1	0	1	1
1	0	1	0	0
1	1	0	0	1

we obtain the binary decision tree



Applying the reduce algorithm produces the desired reduced OBDD:



(c) *answer + explanation*

Yes. We have

$$g(x,y) = \overline{x}y = (x \oplus 1)y = xy \oplus y = y \oplus yx = f(y,x)$$

where the last step uses the algebraic normal form of f computed in part (a).

2 (a) *answer + computation*

Resolution produces the following clauses:

1.	$\{p\}$	
2.	$\{\neg p, q\}$	
3.	$\{\neg q, r\}$	
4.	$\{\neg p, \neg q, \neg r, \neg s\}$	
5.	$\{q\}$	resolve 1, 2, p
6.	$\{\neg q, \neg r, \neg s\}$	resolve 1, 4, p
7.	$\{\neg p, r\}$	resolve 2, 3, q
8.	$\{\neg p, \neg r, \neg s\}$	resolve 2, 4, q
9.	$\{\neg p, \neg q, \neg s\}$	resolve 3, 4, r
10.	$\{r\}$	resolve 3, 5, q
11.	$\{\neg q, \neg s\}$	resolve 3, 6, r
12.	$\{\neg r, \neg s\}$	resolve 5, 6, q
13.	$\{\neg p, \neg s\}$	resolve 5, 9, q
14.	$\{\neg s\}$	resolve 5, 11, q

As there are no further resolvents, the formula is satisfiable.

(b) *answer + explanation*

We first transform φ into an equivalent prenex normal form:

$$\begin{aligned} & \forall x\,(P(x) \rightarrow \exists y\,(Q(x,y) \wedge Q(f(x),g(y)))) \\ \equiv\ & \forall x\,\exists y\,(P(x) \rightarrow Q(x,y) \wedge Q(f(x),g(y))) \end{aligned}$$

Next, we transform the quantifier-free part of the prenex normal form into CNF:

$$\begin{aligned} \equiv\ & \forall x\,\exists y\,(\neg P(x) \vee (Q(x,y) \wedge Q(f(x),g(y)))) \\ \equiv\ & \forall x\,\exists y\,((\neg P(x) \vee Q(x,y)) \wedge (\neg P(x) \vee Q(f(x),g(y)))) \end{aligned}$$

We obtain an equisatisfiable Skolem normal form by replacing the existentially quantified variable y by the fresh Skolem function $h(x)$:

$$\approx\ \forall x\,((\neg P(x) \vee Q(x,h(x))) \wedge (\neg P(x) \vee Q(f(x),g(h(x)))))$$

(c) *answer + explanation*

The following refutation shows that the clausal form is unsatisfiable:

1.	$\{P(x),\, Q(x,y)\}$		
2.	$\{\neg Q(f(x),y)\}$		
3.	$\{\neg P(f(g(x)))\}$		
4.	$\{\neg Q(f(x'),y')\}$	rename 2	$\{x \mapsto x',\, y \mapsto y'\}$
5.	$\{P(f(x'))\}$	resolve 1, 4	$\{x \mapsto f(x'),\, y \mapsto y'\}$
6.	$\square$	resolve 3, 5	$\{x' \mapsto g(x)\}$

In the first resolution step we solve the unification problem

$$\begin{array}{c} \underline{Q(x,y) \approx Q(f(x'),y')} \\ \mathsf{d} \Downarrow \\ \underline{x \approx f(x'),\ y \approx y'} \\ \mathsf{v} \Downarrow \{x \mapsto f(x')\} \\ \underline{y \approx y'} \\ \mathsf{v} \Downarrow \{y \mapsto y'\} \\ \square \end{array}$$

and obtain the mgu $\{x \mapsto f(x'),\, y \mapsto y'\}$. In the second resolution step we solve the unification problem

$$\begin{array}{c} \underline{P(f(g(x))) \approx P(f(x'))} \\ \mathsf{d} \Downarrow \\ \underline{f(g(x)) \approx f(x')} \\ \mathsf{d} \Downarrow \\ \underline{g(x) \approx x'} \\ \mathsf{v} \Downarrow \{x' \mapsto g(x)\} \\ \square \end{array}$$

and obtain the mgu $\{x' \mapsto g(x)\}$.

3 (a) *answer*

For the valuation $v(p) = \mathsf{F}$ and $v(q) = v(r) = \mathsf{T}$, the formula $p \to q \to r$ evaluates to T but $r \to q \to p$ evaluates to F. Hence, $p \to q \to r \not\models r \to q \to p$. Using soundness of natural deduction it follows that the sequent $p \to q \to r \vdash r \to q \to p$ is not valid.

(b) *answer*

1		$\exists x\,(\neg P(x) \land \forall y\,(x = y \to \neg Q(y)))$	premise
2		$\forall x\,(P(x) \lor Q(x))$	assumption
3	x_0	$\neg P(x_0) \land \forall y\,(x_0 = y \to \neg Q(y))$	assumption
4		$\neg P(x_0)$	$\land e_1$ 3
5		$\forall y\,(x_0 = y \to \neg Q(y))$	$\land e_2$ 3
6		$x_0 = x_0 \to \neg Q(x_0)$	$\forall$e 5
7		$x_0 = x_0$	=i
8		$\neg Q(x_0)$	$\to$e 6, 7
9		$P(x_0) \lor Q(x_0)$	$\forall$e 2
10		$P(x_0)$	assumption
11		$\bot$	$\neg$e 10, 4
12		$Q(x_0)$	assumption
13		$\bot$	$\neg$e 12, 8
14		$\bot$	$\lor$e 9, 10 − 11, 12 − 13
15		$\bot$	$\exists$e 1, 3 − 14
16		$\neg\forall x\,(P(x) \lor Q(x))$	$\neg$i 2 − 15

(c) *answer*

1		$\exists x\,(P(x) \wedge \exists y\,(\neg(x = y) \wedge \neg Q(y)))$	premise
2		$\forall x\,(P(x) \wedge Q(x))$	assumption
3	x_0	$P(x_0) \wedge \exists y\,(\neg(x_0 = y) \wedge \neg Q(y))$	assumption
4		$\exists y\,(\neg(x_0 = y) \wedge \neg Q(y))$	$\wedge e_2$ 3
5	y_0	$\neg(x_0 = y_0) \wedge \neg Q(y_0)$	assumption
6		$\neg Q(y_0)$	$\wedge e_2$ 5
7		$P(y_0) \wedge Q(y_0)$	$\forall$ e 2
8		$Q(y_0)$	$\wedge e_2$ 7
9		$\bot$	$\neg$e 8, 6
10		$\bot$	$\exists$e 4, 5−9
11		$\bot$	$\exists$e 1, 3−10
12		$\neg\forall x\,(P(x) \wedge Q(x))$	$\neg$i 2−11

4 (a) *answer + explanation*

From the table

	p	q	$\neg p$	$\mathsf{EX}\,\neg p$	$\mathsf{AF}\,q$	$\mathsf{EX}\,\neg p \vee \mathsf{AF}\,q$	$\mathsf{EG}\,p$	φ
1	✓			✓		✓	✓	✓
2		✓	✓	✓	✓	✓		
3	✓	✓			✓	✓	✓	✓
4	✓				✓	✓	✓	✓

we conclude that the formula φ holds in states 1, 3 and 4.

(b) *answer + explanation*

We have

$\mathcal{M}, 1 \not\models \psi_1$	$\mathcal{M}, 1 \not\models \psi_2$	$\mathcal{M}, 1 \not\models \psi_3$
$\mathcal{M}, 2 \models \psi_1$	$\mathcal{M}, 2 \not\models \psi_2$	$\mathcal{M}, 2 \models \psi_3$
$\mathcal{M}, 3 \not\models \psi_1$	$\mathcal{M}, 3 \models \psi_2$	$\mathcal{M}, 3 \models \psi_3$
$\mathcal{M}, 4 \models \psi_1$	$\mathcal{M}, 4 \models \psi_2$	$\mathcal{M}, 4 \models \psi_3$

i. The formula ψ_1 holds in state 2 of $\mathcal{M}$ because $\neg p$ holds and in state 4 because p holds and q holds in the only possible next state. It does not hold in state 1 due to the path 1^ω and also not in state 3 due to the path $3\,1^\omega$.

ii. The formula ψ_2 holds in state 3 of $\mathcal{M}$ because $p \wedge q$ already holds and in state 4 as state 3 is the only possible next state. It does not hold in state 1 due to the path 1^ω and also not in state 2 due to the path 2^ω.

iii. The formula ψ_3 holds in states 2 and 3 of $\mathcal{M}$, since q holds already in those states. It holds in state 4 as p holds and state 3 is the only possible next state. It does not hold in state 1 due to the path 1^ω.

(c) *answer + explanation*

The formula $\neg p$ holds only in state 2. The formula $\mathsf{AX}(p \wedge q)$ holds only in state 4. Hence we can take $\chi = \neg p \vee \mathsf{AX}(p \wedge q)$.

5	true	false	statement
	☐	☒	The sequent $\neg p \vdash p \vee \neg\neg p$ is valid.
	☐	☒	Every finite LTL model satisfies either $\mathsf{F}\, p$ or $\mathsf{G}\, \neg p$.
	☐	☒	The set of boolean functions $\{0, \bar{}\,, \oplus\}$ is adequate.
	☒	☐	The term $g(f(x), y)$ is free for x in $\exists x\, P(x, g(f(x), y))$.
	☐	☒	The boolean function $f(x, y, z) = x\bar{y} + zy$ is monotone.
	☒	☐	$\mathcal{M}, s \vDash \mathsf{A}[p \,\mathsf{U}\, q] \to (\mathsf{AG}\, p \vee \mathsf{AF}\, q)$ for every model $\mathcal{M}$ and state s.
	☐	☒	Validity of formulas in first-order logic is decidable.
	☐	☒	The Skolem normal form of any valid formula in predicate logic is valid.
	☒	☐	The clause $\{\neg P(a), R(b)\}$ is a resolvent of $\{\neg P(a), Q(x)\}$ and $\{\neg Q(z), R(b)\}$.
	☒	☐	The number of nodes of a reduced OBDD for a boolean function depends on the variable order.