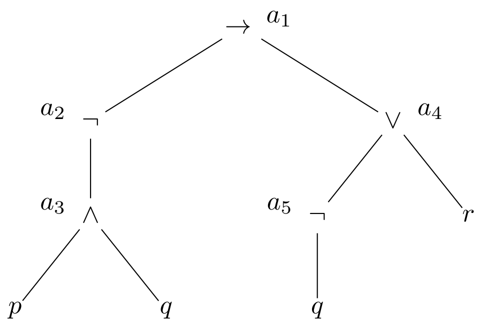
1 (a) *answer + explanation*

The following DPLL derivation results in a satisfying assignment of φ:

$$
\begin{array}{llll}
 & & \| \ \varphi & \\
\Longrightarrow & r^d & \| \ \varphi & \text{(decide)} \\
\Longrightarrow & r^d \ \neg p & \| \ \varphi & \text{(unit propagate)} \\
\Longrightarrow & \neg r & \| \ \varphi & \text{(backtrack)} \\
\Longrightarrow & \neg r \ q^d & \| \ \varphi & \text{(decide)} \\
\Longrightarrow & \neg r \ q^d \ \neg s & \| \ \varphi & \text{(unit propagate)} \\
\Longrightarrow & \neg r \ q^d \ \neg s \ \neg p & \| \ \varphi & \text{(unit propagate)}
\end{array}
$$

(b) *answer + explanation*

We label subformulas of ψ as follows:



Using Tseitin's transformation we obtain

$$\begin{aligned}\psi \approx\ & a_1 \wedge (a_1 \leftrightarrow (a_2 \rightarrow a_4))\\ & \wedge (a_2 \leftrightarrow \neg a_3)\\ & \wedge (a_3 \leftrightarrow p \wedge q)\\ & \wedge (a_4 \leftrightarrow a_5 \vee r)\\ & \wedge (a_5 \leftrightarrow \neg q)\end{aligned}$$

which results in the equisatisfiable CNF

$$\begin{aligned}\psi \approx\ & a_1 \wedge (a_1 \vee a_2) \wedge (a_1 \vee \neg a_4) \wedge (\neg a_1 \vee \neg a_2 \vee a_4)\\ & \wedge (a_2 \vee a_3) \wedge (\neg a_2 \vee \neg a_3)\\ & \wedge (\neg a_3 \vee p) \wedge (\neg a_3 \vee q) \wedge (a_3 \vee \neg p \vee \neg q)\\ & \wedge (a_4 \vee \neg a_5) \wedge (a_4 \vee \neg r) \wedge (\neg a_4 \vee a_5 \vee r)\\ & \wedge (a_5 \vee q) \wedge (\neg a_5 \vee \neg q)\end{aligned}$$

(c) *answer + explanation*

i. Yes, χ is a Horn formula consisting of six Horn clauses. The marking algorithm starts by listing the atoms occuring in χ:

$$q \quad s \quad t \quad r$$

First we mark $\top$. Then we mark q because of the clause $\top \to q$. Next we mark s and r because of the clauses $q \to s$ and $q \to r$. Next we mark t because of the clause $s \wedge r \to t$ and finally we mark $\bot$ because of the clause $s \wedge t \to \bot$. Since $\bot$ is marked, χ is unsatisfiable.

ii. An equivalent CNF is easily obtained from χ:

$$\chi \equiv (\neg q \vee s) \wedge (\neg s \vee \neg t) \wedge (q) \wedge (\neg s \vee \neg r \vee t) \wedge (\neg q \vee r) \wedge (\neg r \vee s)$$

Since not every clause has complementary literals, χ is not valid.

2 (a) *answer + computation*

Resolution produces the following clauses:

1.	$\{\neg P(x),\, Q(x)\}$		
2.	$\{\neg Q(a)\}$		
3.	$\{P(b),\, R(x,y)\}$		
4.	$\{S(x),\, \neg R(a,b)\}$		
5.	$\{\neg S(a)\}$		
6.	$\{\neg P(a)\}$	resolve 1, 2	$\{x \mapsto a\}$
7.	$\{\neg R(a,b)\}$	resolve 4, 5	$\{x \mapsto a\}$
8.	$\{P(b),\, R(u,v)\}$	rename 3	$\{x \mapsto u,\, y \mapsto v\}$
9.	$\{Q(b),\, R(u,v)\}$	resolve 1, 8	$\{z \mapsto b\}$
10.	$\{P(b),\, S(x)\}$	resolve 4, 8	$\{u \mapsto b,\, v \mapsto b\}$
11.	$\{P(b),\, S(z)\}$	rename 10	$\{x \mapsto z\}$
12.	$\{Q(b),\, S(z)\}$	resolve 1, 11	$\{x \mapsto b\}$
13.	$\{P(b)\}$	resolve 5, 10	$\{z \mapsto a\}$
14.	$\{Q(b)\}$	resolve 5, 12	$\{z \mapsto a\}$

As there are no further resolvents (modulo renaming), the formula is satisfiable.

(b) *answer + explanation*

The terms are unifiable:

$$
\begin{array}{c}
g(f(x, f(b,a)), g(x, f(b,y))) \approx g(f(g(y,b), z), g(g(a,b), z)) \\
\mathsf{d} \Downarrow \\
f(x, f(b,a)) \approx f(g(y,b), z),\ g(x, f(b,y)) \approx g(g(a,b), z) \\
\mathsf{d} \Downarrow \\
x \approx g(y,b),\ f(b,a) \approx z,\ g(x, f(b,y)) \approx g(g(a,b), z) \\
\mathsf{v} \Downarrow \{x \mapsto g(y,b)\} \\
f(b,a) \approx z,\ g(g(y,b), f(b,y)) \approx g(g(a,b), z) \\
\mathsf{v} \Downarrow \{z \mapsto f(b,a)\} \\
g(g(y,b), f(b,y)) \approx g(g(a,b), f(b,a)) \\
\mathsf{d} \Downarrow \\
g(y,b) \approx g(a,b),\ f(b,y) \approx f(b,a) \\
\mathsf{d} \Downarrow \\
y \approx a,\ b \approx b,\ f(b,y) \approx f(b,a) \\
\mathsf{v} \Downarrow \{y \mapsto a\} \\
b \approx b,\ f(b,a) \approx f(b,a) \\
\mathsf{t} \Downarrow \\
f(b,a) \approx f(b,a) \\
\mathsf{t} \Downarrow \\
\square
\end{array}
$$

The resulting mgu is

$$\{x \mapsto g(y,b)\}\{z \mapsto f(b,a)\}\{y \mapsto a\} = \{x \mapsto g(a,b),\ y \mapsto a,\ z \mapsto f(b,a)\}$$

(c) *answer + explanation*

We first eliminate the implication:

$$
\begin{aligned}
&\forall x\, \exists y\, \forall x\ z\big(R(x,y,z) \rightarrow \exists w\, S(y,z,w)\big) \\
&\equiv\ \forall x\, \exists y\, \forall z\, \big(\neg R(x,y,z) \lor \exists w\, S(y,z,w)\big)
\end{aligned}
$$

Next, we bring all quantifiers to the front to obtain a prenex normal form:

$$\equiv\ \forall x\, \exists y\, \forall z\, \exists w\, \big(\neg R(x,y,z) \lor S(y,z,w)\big)$$

We obtain an equisatisfiable Skolem normal form by replacing the existentially quantified variables by Skolem functions, y by $f(x)$ and w by $g(x,z)$:

$$\approx\ \forall x\, \forall z\, \big(\neg R(x, f(x), z) \lor S(f(x), z, h(x,z))\big)$$

3 (a)

answer

The sequent $p,\ r \to \neg p,\ \neg r \wedge s \to t,\ \neg t \vdash \neg s$ is valid:

1	p	premise
2	$r \to \neg p$	premise
3	$\neg r \wedge s \to t$	premise
4	$\neg t$	premise
5	s	assumption
6	$\neg\neg p$	$\neg\neg$i 1
7	$\neg r$	MT 2, 6
8	$\neg r \wedge s$	$\wedge$i 7, 5
9	t	$\to$e 3, 8
10	$\bot$	$\neg$e 9, 4
11	$\neg s$	$\neg$i 5 − 10

(b)

answer

The sequent $\forall x\,(x = a \vee x = b),\ \forall x\,(x = a \to P(x)),\ \forall x\,(x = b \to Q(x)) \vdash \forall x\,(P(x) \vee Q(x))$ is valid:

1		$\forall x\,(x = a \vee x = b)$	premise
2		$\forall x\,(x = a \to P(x))$	premise
3		$\forall x\,(x = b \to Q(x))$	premise
4	x_0	$x_0 = a \vee x_0 = b$	$\forall$e 1
5		$x_0 = a$	assumption
6		$x_0 = a \to P(x_0)$	$\forall$e 2
7		$P(x_0)$	$\to$e 6, 5
8		$P(x_0) \vee Q(x_0)$	$\vee$i$_1$ 7
9		$x_0 = b$	assumption
10		$x_0 = b \to Q(x_0)$	$\forall$e 3
11		$Q(x_0)$	$\to$e 10, 9
12		$P(x_0) \vee Q(x_0)$	$\vee$i$_2$ 11
13		$P(x_0) \vee Q(x_0)$	$\vee$e 4, 5 − 8, 9 − 12
14		$\forall x\,(P(x) \vee Q(x))$	$\forall$i 4 − 13

(c) *answer*

The sequent $\forall x\,(x = a \vee x = b), \forall x\,(x = a \to P(x)), \forall x\,(x = b \to Q(x)) \vdash \forall x\,P(x) \vee \forall x\,Q(x)$ is not valid. Take the model $\mathcal{M}$ with the universe $A = \{0, 1\}$ and the following interpretations:

$$a^{\mathcal{M}} = 0 \qquad b^{\mathcal{M}} = 1 \qquad P^{\mathcal{M}} = \{0\} \qquad Q^{\mathcal{M}} = \{1\}$$

We have $\mathcal{M} \vDash \forall x\,(x = a \vee x = b)$, $\mathcal{M} \vDash \forall x\,(x = a \to P(x))$ and $\mathcal{M} \vDash \forall x\,(x = b \to Q(x))$ but $\mathcal{M} \nvDash \forall x\,(P(x) \vee \forall x\,Q(x))$. Hence by soundness of natural deduction the sequent is not valid.

4 (a) *answer + explanation*

From the table

	p	q	$\mathsf{EF}\,p$	$\mathsf{AX}\,q$	$\mathsf{A}[\mathsf{EF}\,p\,\mathsf{U}\,\mathsf{AX}\,q]$	φ
1	✓		✓			
2		✓	✓			
3	✓		✓	✓	✓	✓
4		✓	✓			✓

we conclude that the CTL formula $\varphi = \mathsf{EF}\,\mathsf{A}[\mathsf{EF}\,p\,\mathsf{U}\,\mathsf{AX}\,q]$ holds in states 3 and 4.

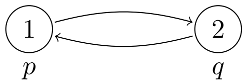
(b) *answer + explanation*

For $\psi = q \wedge \mathsf{F}(\mathsf{G}\,p \vee \mathsf{G}\,q)$ we have $\mathcal{M}, s \vDash \psi$ if and only if $s = 2$: If a path starts in 2, it is either 2^ω or it is of the form $2^n\,1^\omega$ for some $n > 0$. Hence, $\mathcal{M}, 2 \vDash \psi$. On the other hand, $\mathcal{M}, 1 \nvDash \psi$ and $\mathcal{M}, 3 \nvDash \psi$ because q does not hold in the respective states. Finally, the path $(4\,3)^\omega$ establishes $\mathcal{M}, 4 \nvDash \psi$.

There are many other solutions. For instance, $\psi = q\,\mathsf{W}\,(\mathsf{G}\,p)$.

(c) *answer + explanation*

Consider the following model $\mathcal{M}$:



From $\mathcal{M}, 1 \models p$ and $\mathcal{M}, 2 \models q$ we obtain $\mathcal{M}, 1 \models \mathsf{E}_{\{p,q\}}\, \mathsf{G}\, \top$. However, $\mathcal{M}, 1 \not\models \mathsf{E}_{\{p \wedge q\}}\, \mathsf{G}\, \top$ as no path satisfies the fairness constraint $p \wedge q$.

5	true	false	statement
	X		Austria is 3-colorable.
	X		The formulas $p \wedge \neg q$ and $p \wedge q$ are equisatifiable.
		X	The proof rule $\neg\neg$e is a derived rule of natural deduction.
	X		The clause $\{P(z)\}$ is a resolvent of $\{\neg P(x)\}$ and $\{P(z), P(y)\}$.
	X		If a binary function f is monotone and $f(1,1) = 0$ then $f(x,y) = 0$.
	X		The set $\{\mathsf{R}, \mathsf{U}, \mathsf{X}\}$ is an adequate set of temporal connectives for LTL.
		X	Deciding the satisfiability of propositional Horn formulas is NP-complete.
	X		The problem whether an arbitrary propositional formula is valid is decidable.
		X	The sequent $\forall x\,\exists y\,P(x,y), \forall x\,\forall y\,(P(x,y) \rightarrow Q(x,y)) \vdash \exists y\,\forall x\,Q(x,y)$ is valid.
		X	Every reduced OBDD for an n-ary boolean function has at most $2^{n+1} - 1$ nodes.