Exercises.

9.0 Study Chapter 5.9

9.1 Exercise 5.3.4

9.2 Exercise 5.3.5

9.3 Exercise 5.4.1

9.4 Exercise 5.4.2

Optional Exercises.

Set $\mathbf{R} = \{P, Q\}$, where P, Q are unary; $\mathbf{F} = \{+, -, \cdot\}$, where $+, -, \cdot$ are binary and $\mathbf{C} = \emptyset$. Let $\mathsf{L} = \mathsf{L}(\mathbf{R}, \mathbf{F}, \mathbf{C})$ be fixed in the following. Define the *size* $\mathsf{size}(t)$ of a term t of L by structural recursion:

$$\begin{aligned}
\mathsf{size}(x) &:= 1 \qquad x \text{ a variable} \\
\mathsf{size}(t_1 + t_2) &:= \mathsf{size}(t_1) + \mathsf{size}(t_2) + 1 \\
\mathsf{size}(t_1 - t_2) &:= \mathsf{size}(t_1) + \mathsf{size}(t_2) + 1 \\
\mathsf{size}(t_1 \cdot t_2) &:= \mathsf{size}(t_1) + \mathsf{size}(t_2) + 1
\end{aligned}$$

1. What is the size of the term $((x + y) - z) \cdot ((u - (v - w)) \cdot (x - (u + w)))$, where $x, y, z, u, v, w \in \mathbf{V}$?

2. Prove that for all terms s, t the term s is not a proper initial segment of t. Use induction on the size of t.

3. Compare the previous proof to the proof of Exercise 2.2.5.

4. Generalise the definition of size to formulas, such that

$$\mathsf{size}((P(x) \wedge Q((u + w)))) = 4 \ .$$

   (NB: There need not be a unique solution.)

5. Prove for all formulas A, B of L the formula A is not a proper initial segment of B. Either use induction on B or induction on the size of B.

6. State the unique parsing theorem for first-order formulas formally.

7. Prove the unique parsing theorem for first-order formulas.

8. Is the following claim correct?

   Claim: In any set of h cats, all cats have the same colour.

   "*Proof*":

   Base: For $h = 1$. In any set containing just one cat, all cats clearly have the same colour.

   Step: For $k \geq 1$ assume that the claim is true for $h = k$ and prove that it is true for $h = k+1$. Take any set H of $k+1$ cats. Remove one cat from this set to obtain the set H_1 with just k cats. By induction hypothesis (IH), all cats in H_1 have only one colour. Now replace the removed cat, remove a different one from H to obtain a set H_2. By the same argument all cats in H_2 have the same colour. Therefore all the cats in H have to have the same colour and the proof is complete.