Exercises.

1.0 Study Chapter 1 in "First-Order Logic and Automated Theorem Proving" (henceforth "the book").

1. Consider the symbols employed in Chapter 1 to describe the mathematical theory *arithmetic*.

   1.1 Express formally that every natural number is either even or odd.

   1.2 Express formally that for every natural number there exists a larger one. You may use the binary relation symbol $<$, expressing the "greater than" relation.

1.3 What is the truth value of the sentences

$$\forall x \exists y (x > y) \quad \text{and} \quad (\forall x \exists y (x > y)) \land \neg(\forall x \exists y (x > y)) ,$$

when interpreted in arithmetic. Explain your answer.

1.4 Give a precise definition of the liveness property and show that the protocol from the lecture does not fulfil the liveness property.

Solution. Liveness can be defined as follows: for every reachable request-state and for every path starting at this state there is a future state (on this path) which is a critical state. Obviously the path $s_1, s_4, s_7, s_1, \ldots$ starting at the state s_1, where the first process tries to enter its critical section does not fulfil this property. □

1.5 Can you express the liveness property using the introduced connectives and quantifiers?

Solution. A simple yes/no answer is not possible, as we have not (yet) defined precisely how connectives and quantifiers are to be interpreted. Thus, we collect some observations.

In order to express the liveness property one has to *quantify* over all *paths* starting at a certain state. To express this using the introduced connectives and quantifiers, we would like to write (for process 1):

$$\forall x (P(k_0, x) \land T_1(x) \to \forall Q(\text{path}(Q) \to (\exists y Q(x, y) \land C_1(y)))) .$$

where path(Q) asserts that the relation Q expresses a path in the protocol.

Note that first-order logic is *not* expressive enough for this formulas. Firstly, in the above formula we use the *second-order quantor* $\forall Q$, i.e. we quantify over a predicate instead of a variable. In first-order logic we quantify only over variables. Secondly we have to *define* the notion of *path* in some way which again is not expressible in first-order logic. This is only possible by extending first-order logic by set theory or using second-order logic. The reason for this will become clear when the so-called *Compactness Theorem* is discussed.

Thus the formula above is not a first-order formula. One may try whether there is another way to express the liveness property in first-order logic. However, again one can prove that this is impossible. □