Exercises.

10.0 Study Chapter 5.9, 6.1–6.4

10.1 Exercise 5.9.2

Solution. The idea of the proof is to use the Compactness Theorem 5.9.1. We start by representing an arbitrary but fixed graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ as a first-order formula. Define $\mathsf{L} = \mathsf{L}(\mathbf{R}, \mathbf{F}, \mathbf{C})$ by setting

$$\begin{aligned}
&\mathbf{R} := \{E, \mathrm{eq}, R, G, B, Y\} && E, \mathrm{eq} \text{ binary}, R, G, B, Y \text{ unary}\\
&\mathbf{F} := \emptyset\\
&\mathbf{C} := \{c_v \mid v \in \mathbf{V}\}
\end{aligned}$$

Furthermore we define the set of formulas S, s.t. S includes

$$(\forall x)\ x \ \mathrm{eq}\ x \tag{1}$$
$$(\forall x, y)\ x \ \mathrm{eq}\ y \to y \ \mathrm{eq}\ x \tag{2}$$
$$(\forall x, y, z)\ x \ \mathrm{eq}\ y \wedge y \ \mathrm{eq}\ z \to x \ \mathrm{eq}\ z\ , \tag{3}$$

and

$$E(c_v, c_w) \qquad \text{for all } (v, w) \in \mathbf{E} \tag{4}$$
$$(\forall x)\ \neg E(x, x) \wedge (\forall x, y)\ E(x, y) \to E(y, x)\ , \tag{5}$$

and

$$\neg(c_v \ \mathrm{eq}\ c_w) \qquad \text{for all } v, w \in \mathbf{V} \tag{6}$$

Here the formulas (1)–(3) express that the binary relation eq is an equivalence relation, the formulas (4)–(5) express the edges in $\mathbf{G}$ and (6) guarantees that different constants c_v, c_w are interpreted by different vertices $v, w \in \mathbf{V}$. Obviously $\mathbf{G}$ can be extended to a model of L (by interpreting eq as the identity) and satisfies S. Moreover it is not hard to argue that any model $\mathbf{M}$ of S is a graph that has the same structure as $\mathbf{G}$.[1]

[1] More precisely $\mathbf{M}$ is *isomorph* to $\mathbf{G}$.

To conclude the construction we extend S by the formula X:

$$(\forall x)(R(x) \vee G(x) \vee B(x) \vee Y(x)) \wedge (\forall x, y)(E(x,y) \rightarrow ((R(x) \rightarrow \neg R(y)) \wedge \\ \wedge (G(x) \rightarrow \neg G(y)) \wedge (B(x) \rightarrow \neg B(y)) \wedge (Y(x) \rightarrow \neg Y(y)))) \ .$$

Clearly $S \cup \{X\}$ is satisfiable iff G is four colorable. Moreover, let S' denote the finite set $S' = S_0 \cup \{X\} \subseteq S \cup \{X\}$. Then S' is satisfiable iff a finite subgraph of G is four colorable. By assumption this implies that all S' are satisfiable. In a similar way, we see that all finite subsets S' of $S \cup \{X\}$ are satisfiable. Hence by Theorem 5.9.1 $S \cup \{X\}$ is satisfiable. Which in turn implies that G is four colorable.

This completes the proof that if any finite subgraph of G is four colorable, then G is four colorable. $\square$

10.2 Exercise 6.1.1

10.3 Exercise 6.2.1

Optional Exercises.

1. Let $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ be a finite directed graph. We write $\text{path}(v, w)$ to indicate that there exists a path from v to w in $\mathbf{G}$. Then no first-order formulas $X(x, y)$ can exists, such that $X(x, y)$ is true in $\mathbf{G}$ for some assignment $\mathbf{A}$ iff $\text{path}(x^{\mathbf{A}}, y^{\mathbf{A}})$ holds. I.e. reachability is not first-order definable.

2. Let $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ be defined as above. There exists a second-order formula $\text{path}(P)$, expressing that P is path in $\mathbf{G}$.