Exercises.

2.0 Review your studies on induction on numbers (high school textbooks, Formale Methoden I & II) and Theorem 2.2.3 & 2.2.4 in the book on p. 11–12.

2.1 Show $R \cup (S \cap T) = (R \cup S) \cap (R \cup T)$ formally. (The review of the definitions of union and intersection are part of the exercise.)

Solution.

$$\begin{aligned} R \cup (S \cap T) &= \{x \mid x \in R \text{ or } x \in (S \cap T)\} = \\ &= \{x \mid x \in R \text{ or } (x \in S \text{ and } x \in T)\} \\ &= \{x \mid (x \in R \text{ or } x \in S) \text{ and } (x \in R \text{ or } x \in T)\} \\ &= (R \cup S) \cap (R \cup T)\ . \end{aligned}$$

□

2.2 Show by induction on n:

$$\text{for all } n \geq 4 \colon n^2 \geq 2n + 1\ .$$

2.3 Show by induction on n:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}\ .$$

2.4 Show by structural induction: Each expression (as defined in the lecture) contains exactly as many left as right brackets.

Solution.

Base: If the expression is a number or a letter, then the assertion holds trivially.

Step: We consider an expression E.

induction hypothesis (IH): The assertion holds for all proper subexpressions of E.

Case-distinction:

a) Case: E has the form $E_1 + E_2$, where IH holds for E_1 and E_2. Thus by IH the number of left and right brackets is equal for E_1 and E_2. Hence the same holds for E.

b) Case: E has the form $E_1 \cdot E_2$; as in case a).

c) Case: E has the form (E_1), where IH holds for E_1. Thus by IH the number of left and right brackets is equal for E_1. Suppose n is the number of left (or right) brackets in E_1. Hence

$$\text{number of left brackets in } (E_1) = n + 1 = \text{number of right brackets in } (E_1) \; .$$

□

2.5 Show the missing base cases and step-cases in the mutual induction proof in the lecture.

Solution. Since induction on natural numbers is applied one has to show that both statements hold for n equals 0 (base case) and under the assumption that both statements hold for n they also hold for n+1 (step case).
The former reduces to the statements "The automaton is off after 0 steps iff 0 is even." and "The automaton is on after 0 steps iff 0 is odd." which both evaluate to *true* since both sides of the equivalences are *true* or *false* respectively. The latter requires to show "The automaton is off after n+1 steps iff n+1 is even." and "The automaton is on after n+1 steps iff n+1 is odd." where one may assume "The automaton is off after n steps iff n is even." and "The automaton is on after n steps iff n is odd.". Since the automaton is off after n+1 steps if and only if it is on after n steps (according to its description) which is if and only if n is odd (by assumption) which again is if and only if n+1 is even (trivial) the first statement holds, the prove of the second statement goes analogously. □

Optional Exercises.

1. Exercise 2.2.1
2. Exercise 2.2.2
3. Exercise 2.2.3