
Exercises.

3.0 Study Chapter 2, sections 2.1–2.4

3.1 Exercise 2.2.1

Solution. Base: Atomic formulas contain neither binary symbols nor parentheses, so the claim trivially holds.

Step: Assume the claim holds for a (pair of) propositional formula(s) then it also holds for the negation/composition of this (pair of) formula(s) since in the former case only a unary symbol and no parentheses and in the latter case exactly one binary symbol and one left and one right parenthesis are added. □

3.2 Exercise 2.2.3

Solution. Base: We have to show that for every atomic formula X, $X \in \mathbf{P}^\circ$. By assumption the removed formula F is not atomic. The base case holds.

Step: Suppose X has the form $\neg X'$ or $(X_1 \circ X_2)$ for propositional formulas X', X_1, and X_2.

IH: The formulas X', X_1, and X_2 are in $\mathbf{P}^\circ$. By assumption the removed formula F is no negated propositional formula and not a composed pair of propositional formulas. Hence F is different form $\neg X'$ and $(X_1 \circ X_2)$. This implies that $\neg X' \in \mathbf{P}^\circ$ ($(X_1 \circ X_2) \in \mathbf{P}^\circ$). This completes the step case.

We have shown that for any propositional formula $X \in \mathbf{P}$, $X \in \mathbf{P}^\circ$ holds. This implies that $\mathbf{P} \subseteq \mathbf{P}^\circ$ but $\mathbf{P}^\circ = \mathbf{P} - \{F\}$ which is impossible. We have derived a contradiction to the existence of F. □

3.3 Exercise 2.2.4

Solution. Base: Obviously an atomic formula has no proper initial segment, hence the base case follows trivially. Step: If X is a composed formula $(X_1 \circ X_2)$ and Y a proper initial segment of X, then one of the following cases holds:

- $Y = ($
- $Y = (Y'$ and Y' is an initial segment of X_1 (not necessarily proper).
- $Y = (X_1 \circ Y'$ and Y' is an initial segment of X_2 (not necessarily proper).
- $Y = (X_1 \circ$.

In the cases two and three we see either by IH or Exercise 2.2.1 that $l(Y') \geq r(Y')$. Hence in all cases $l(Y) > r(Y)$. $\square$

3.4 Exercise 2.2.5

Solution. By Exercise 2.2.1 every propositional formula has the same number of left as right parentheses and by Exercise 2.2.4 every proper initial segment of a restricted formula has more left than right parentheses. Hence no proper initial segment of a restricted formula can be itself a propositional (and hence restricted) formula. $\square$

3.5 Exercise 2.2.6

Solution. Assume X is not identical with U then we can assume without loss of generality (w.l.o.g.) that the restricted formula X is a proper initial segment of the restricted formula U. However this contradicts Exercise 2.2.5. Hence X is identical with U. This implies that the two binary symbols must be the same and that Y is identical with V. $\square$

Optional Exercises.

1. Exercise 2.2.7

2. Exercise 2.2.8