Exercises.

4.1 Exercise 2.4.1

Solution. We show something slightly stronger: Let f denote a *partial* mapping from the set of propositional letters to $\mathbf{Tr}$. Then we claim the existence of a Boolean valuation v, such that $v(P) = f(P)$ for all propositional letters P in the domain of f.

Define a function $v\colon \mathbf{P} \to \mathbf{Tr}$ by structural recursion:

- Base: Set

$$v(\top) := \mathbf{t} \qquad v(\bot) := \mathbf{f}$$
$$v(P) := \begin{cases} f(P) & P \text{ is in the domain of } f \\ \mathbf{f} & \text{otherwise} \end{cases}$$

- Step: Set

$$v(\neg X) := \neg v(X)$$
$$v(X \circ Y) := v(X) \circ v(Y)$$

By the Principle of Structural Recursion (Thm. 2.2.4) the function v is unique. By definition of v we have:

$$v(P) = f(P) \qquad P \text{ is in the domain of } f\ .$$

Finally by definition of a Boolean valuation (Def. 2.4.1) v is a Boolean valuation. Hence the claim follows. □

4.2 Exercise 2.4.2

Solution. Let $v_1\colon \mathbf{P} \to \mathbf{Tr}$, $v_2\colon \mathbf{P} \to \mathbf{Tr}$ denote two different Boolean valuations such that for all propositional letters $P \in S$:

$$v_1(P) = v_2(P) \tag{$\dagger$}$$

We claim:

$$v_1(X) = v_2(X) \qquad \text{for all propositional formulas } X \text{ such that } X \text{ contains only propositional letters in } S. \tag{$\star$}$$

We show the claim by structural induction on X. We use the format of structural induction as expressed in Thm. 2.6.3.

- BASE: We have to show the property ($\star$) for every atomic formula and its negation.
  Suppose X is a propositional letter P. By assumption X contains only propositional letters from S, hence $P \in S$. Thus $v_1(X) = v_2(X)$ follows from (†).
  Now consider the case where $X = \neg P$. Thus
  $$v_1(X) = v_1(\neg P) = \neg v_1(P) = \neg v_2(P) = v_2(\neg P) = v_2(X) \; ,$$
  follows by one application of (†).
  Finally consider the case where X is propositional constant or its negation. Then the claim is trivially true.
- STEP: We have to consider the cases (i) $X = \neg\neg X_1$, (ii) X an α-formula and (iii) X a β-formula. By induction hypothesis (IH) property ($\star$) holds for X_1, $\alpha_1, \alpha_2, \beta_1, \beta_2$.
  CASE (i): Then
  $$v_1(X) = v_1(\neg\neg X_1) = \neg\neg v_1(X_1) = \neg\neg v_2(X_1) = v_2(\neg\neg X_1) = v_2(X) \; ,$$
  follows by definition of a Boolean valuation and IH.
  CASE (ii): Thus
  $$\begin{aligned} v_1(\alpha) &= v_1(\alpha_1) \wedge v_1(\alpha_2) && \text{Proposition 2.6.1} \\ &= v_2(\alpha_1) \wedge v_2(\alpha_2) && \text{IH} \\ &v_2(\alpha) \; . \end{aligned}$$
  CASE (iii): Similar to case (ii).

□

4.3 Exercise 2.4.4

Solution. We only show Exercise 2.4.4.1, the two other cases are similar:

4.3.1 The following sequence of equivalences follows by the definition of the Boolean valuation v and the definition of mapping $\equiv : \mathbf{Tr} \to \mathbf{Tr}$:
$$v(X \equiv Y) = \mathbf{t} \iff (v(X) \equiv v(Y)) = \mathbf{t} \iff v(X) = v(Y) \; .$$

□

Optional Exercises.

1. Exercise 2.2.7
2. Exercise 2.2.8
3. Exercise 2.4.3
4. Exercise 2.4.5