**Exercises.**

5.0 Study Chapter 3.1–3.4

5.1 Exercise 3.1.1

5.2 Exercise 3.3.1

5.3 Exercise 3.4.2

*Solution.* Let $\mathbf{R}$ denote a satisfiable resolution expansion. Let $\mathbf{R}^*$ denote the resolution expansion resulting from an application of a resolution expansion rule or the resolution rule to $\mathbf{R}$. We have to show that $\mathbf{R}^*$ is satisfiable.

We do a case-analysis depending on whether a resolution expansion rule or the resolution rule has been applied to $\mathbf{R}$.

Case: A resolution expansion rule has been applied to a disjunction $D$ in the resolution expansion $\mathbf{R}$. Let $X$ denote the non-literal, the expansion rule was applied to. We only consider the subcase, where $X = \alpha$. We write $[\alpha, X_1, \ldots, X_n]$ for $D$ to indicate the occurrence of $\alpha$. By definition $\mathbf{R}^*$ is extended by the disjunctions $[\alpha_1, X_1, \ldots, X_n]$ and $[\alpha_2, X_1, \ldots, X_n]$.

By assumption there exists a Boolean valuation $v$, such that $v([\alpha, X_1, \ldots, X_n]) = \mathbf{t}$. W.l.o.g we assume that $v(\alpha) = \mathbf{t}$. Hence, by Prop. 2.6.1 $v(\alpha_1) = v(\alpha_2) = \mathbf{t}$ and thus $v([\alpha_1, X_1, \ldots, X_n]) = v([\alpha_2, X_1, \ldots, X_n]) = \mathbf{t}$. It follows that $\mathbf{R}^*$ is satisfiable.

Case: The resolution rule has been applied to two disjunctions $D_1$ and $D_2$ in $\mathbf{R}$. To simplify notation, we assume (w.l.o.g.) that $D_1 = [X, X_1, \ldots, X_n]$ and $D_2 = [\neg X, X_1, \ldots, X_n]$. By assumption $v(D_1) = v(D_2) = \mathbf{t}$. The resolvent of $D_1$ and $D_2$ has the form $[X_1, \ldots, X_n, X_1, \ldots, X_n]$. As $v(X) \neq v(\neg X)$ for at least one $i$, $v(X_i) = \mathbf{t}$. It follows that $v([X_1, \ldots, X_n, X_1, \ldots, X_n]) = \mathbf{t}$ and hence $\mathbf{R}^*$ is satisfiable. □

5.4 Exercise 3.4.3

*Solution.* We prove the claim, by assuming the existence of a closed resolution expansion $\mathbf{R}$ for satisfiable $S$. As $\mathbf{R}$ is a resolution expansion of $S$ it is obtained from the initial resolution expansion of $S$ by a sequence of resolution expansion or resolution rules. By Exercise 5.3, $\mathbf{R}$ is thus satisfiable. But this is a contradiction to the fact that $\mathbf{R}$ contains the empty clause. □

5.5 Exercise 3.4.4

*Solution.* A resolution proof of $X$ is closed resolution expansion for $\{\neg X\}$. By Exercise 5.4, $\{\neg X\}$ is not satisfiable. Hence for all Boolean valuations $v$, $v(X) = \mathbf{t}$ holds. Therefore $X$ is a tautology. $\square$

**Optional Exercises.**

1. Exercise 3.1.2
2. Exercise 3.2.1
3. Exercise 3.2.2
4. Exercise 3.3.3