Exercises.

6.0 Study Chapter 3.5–3.9

6.1 Exercise 3.6.1

Solution. To show the claim, we follow the hint. Let $\mathcal{C}$ be a propositional consistency property; let $\mathcal{C}^+$ consist of all subsets of members of $\mathcal{C}$. Cleary $\mathcal{C}^+$ extends $\mathcal{C}$ and it is easy to see that it is subset closed. We show that $\mathcal{C}^+$ is a consistency property by checking the 5 conditions. Let $S \in \mathcal{C}^+$.

Case For any propositional letter A, not both $A \in S$ and $\neg A \in S$: Suppose $\{A, \neg A\} \subseteq S$. By definition S is a subset of a set $S' \in \mathcal{C}$. Hence $\{A, \neg\} \subseteq S'$, which shows that $\mathcal{C}$ is not a propositional consistency property. Contradiction to the assumption that both $A \in S$ and $\neg A \in S$.

Case $\bot \notin S$, $\neg\top \notin S$: Similar to case 1.

Case $\neg\neg Z \in S$ implies $S \cup \{Z\} \in \mathcal{C}^+$: Reasoning as in case 1), we conclude the existence of a set $S' \in \mathcal{C}$, such that $S' \cup \{Z\} \in \mathcal{C}$. As $\mathcal{C}^+$ consist of all subsets of members of $\mathcal{C}$, $S \cup \{Z\} \in \mathcal{C}^+$, too.

Case $\alpha \in S$ implies $S \cup \{\alpha_1, \alpha_2\} \in \mathcal{C}^+$: As in case 3).

Case $\beta \in S$ implies $S \cup \{\beta_1\} \in \mathcal{C}^+$ or $S \cup \{\beta_2\} \in \mathcal{C}^+$: As in case 3). □

6.2 Exercise 3.6.2

Solution. Let $\mathcal{C}$ denote a propositional consistency property of finite character. We have to show that, if $S \in \mathcal{C}$ and $S' \subseteq S$, then $S' \in \mathcal{C}$. We fix S and S'.

Assume S' is finite, then $S' \in \mathcal{C}$ follows by the assumption that $\mathcal{C}$ is of finite character. Assume otherwise that S' is infinite. To show $S' \in \mathcal{C}$, we show that for every finite $S_0 \subseteq S'$, $S_0 \in \mathcal{C}$. Then $S' \in \mathcal{C}$ follows by definition of finite character.

To show $S_0 \subseteq S'$ for any finite subset S_0 of S', it suffices to realise that any such subset S_0 is also a (finite) subset of S, hence $S_0 \in \mathcal{C}$, by the assumption that $S \in \mathcal{C}$. □

6.3 Exercise 3.6.3

Solution. To show the claim, we follow the hint. Let $\mathcal{C}$ be a propositional consistency property that is subset closed; let $\mathcal{C}^*$ consist of those sets, all of whose finite subsets are in $\mathcal{C}$. Cleary $\mathcal{C}^*$ extends $\mathcal{C}$ and $\mathcal{C}^*$ is of finite character. We show that $\mathcal{C}^*$ is a consistency property by checking the 5 conditions. Let $S \in \mathcal{C}^*$. We only consider two interesting cases.

Case For any propositional letter A, not both $A \in S$ and $\neg A \in S$: Suppose $\{A, \neg A\} \subseteq S$. By assumption $S \in \mathcal{C}^*$, hence any finite subset of S is a member of $\mathcal{C}$. In particular $\{A, \neg A\} \in \mathcal{C}$, which contradicts the assumption that $\mathcal{C}$ is a propositional consistency property.

Case $\alpha \in S$ implies $S \cup \{\alpha_1, \alpha_2\} \in \mathcal{C}^*$: We show that for any finite $S_0 \subseteq S$, $S_0 \cup \{\alpha_1, \alpha_2\} \in \mathcal{C}$. Note that $S_0 \in \mathcal{C}$ by definition of $\mathcal{C}^*$.

We consider the following subcases (i) $\alpha \in S_0$ and (ii) $\alpha \notin S_0$. In subcase (i), we immediately get $S_0 \cup \{\alpha_1, \alpha_2\} \in \mathcal{C}$ by the assumption that $\mathcal{C}$ is a consistency property. Now consider subcase (ii). As S_0 is finite, $S_0 \cup \{\alpha\}$ is finite, too and $S_0 \cup \{\alpha\} \subseteq S$ by assumption. Hence $S_0 \cup \{\alpha\} \in \mathcal{C}^*$ as $\mathcal{C}^*$ is subset closed. Thus $S_0 \cup \{\alpha\} \in \mathcal{C}$ by definition of $\mathcal{C}^*$. Hence $S_0 \cup \{\alpha, \alpha_1, \alpha_2\} \in \mathcal{C}$. Finally we apply the assumption that $\mathcal{C}$ is subset closed to conclude that $S_0 \cup \{\alpha_1, \alpha_2\} \in \mathcal{C}$. □

6.4 Exercise 3.6.6

Optional Exercises.

1. We call a set M *countable* if it is not finite and if there is a surjective map of the natural numbers $\mathbb{N}$ onto M. Show that, if the list of propositional letters is countable, the entire set of propositional formulas is countable as well.
2. Exercise 3.6.4
3. Exercise 3.6.5
4. Exercise 3.8.1
5. Exercise 3.8.2