Exercises.

7.0 Study Chapter 5.1–5.3

7.1 Exercise 5.1.1

7.2 Suppose $\mathbf{M} = (\mathbf{D}, \mathbf{I})$ is a model for L, $\mathbf{A}$ an assignment in $\mathbf{M}$, σ is a substitution. Define $\mathbf{B}$ by setting for each variable $v^{\mathbf{B}} = (v\sigma)^{\mathbf{I},\mathbf{A}}$. Then $t^{\mathbf{I},\mathbf{B}} = (t\sigma)^{\mathbf{I},\mathbf{A}}$ for any term t.

Solution. We prove the claim by induction on t.

BASE: t is constant $c \in \mathbf{C}$. Then $c^{\mathbf{I},\mathbf{B}} = c^{\mathbf{I}} = (c\sigma)^{\mathbf{I}} = (c\sigma)^{\mathbf{I},\mathbf{A}}$ follows easily. Now assume t is a variable $x \in \mathbf{V}$. Then $x^{\mathbf{I},\mathbf{B}} = x^{\mathbf{B}} = (x\sigma)^{\mathbf{I},\mathbf{A}}$.

STEP: t is a complex term $f(t_1, \ldots, t_n)$. Then

$$\begin{aligned}
[f(t_1, \ldots, t_n)]^{\mathbf{I},B} &= f^{\mathbf{I}}(t_1^{\mathbf{I},B}, \ldots, t_n^{\mathbf{I},B}) && \text{definition of value} \\
&= f^{\mathbf{I}}((t_1\sigma)^{\mathbf{I},A}, \ldots, (t_n\sigma)^{\mathbf{I},A}) && \text{IH} \\
&= [f(t_1\sigma, \ldots, t_n\sigma)]^{\mathbf{I},A} && \text{definition of substitution}
\end{aligned}$$

□

7.3 Exercise 5.3.1

Solution. We prove the (stronger) proposition: $[X\{x \mapsto t\}]^{\mathbf{I},\mathbf{B}} = X^{\mathbf{I},\mathbf{A}}$ for some x-variant $\mathbf{B}$ of $\mathbf{A}$ by induction on X. As preparation, we prove the following claim. We set $\sigma = \{x \mapsto t\}$.

Claim. *For all terms s, $(s\sigma)^{\mathbf{I},\mathbf{B}} = s^{\mathbf{I},\mathbf{A}}$.*

Proof of the Claim. BASE: s is constant $c \in \mathbf{C}$. Then $(c\sigma)^{\mathbf{I},\mathbf{B}} = c^{\mathbf{I},B} = c^{\mathbf{I}} = c^{\mathbf{I},\mathbf{A}}$ holds. Now assume s is a variable $y \in \mathbf{V}$. We assume $y = x$, otherwise the claim follows as $\mathbf{B}$ is an x-variant of $\mathbf{A}$. Then $(x\sigma)^{\mathbf{I},\mathbf{B}} = t^{\mathbf{I},\mathbf{B}} = t^{\mathbf{I}} = t^{\mathbf{I},\mathbf{A}}$ holds. The last equality holds, as t is closed.

STEP: s is a complex term $f(s_1, \ldots, s_n)$. Then

$$\begin{aligned}
[f(s_1, \ldots, s_n)\sigma]^{\mathbf{I},B} &= f^{\mathbf{I}}((s_1\sigma)^{\mathbf{I},B}, \ldots, (s_n\sigma)^{\mathbf{I},B}) \\
&= f^{\mathbf{I}}(s_1^{\mathbf{I},A}, \ldots, s_n^{\mathbf{I},A}) && \text{IH} \\
&= [f(s_1, \ldots, s_n)]^{\mathbf{I},A}
\end{aligned}$$

This establishes the claim. □

We proceed with the proof of the proposition.

BASE: Suppose X is atomic, s.t. $X = P(s_1, \ldots, s_n)$. By the previous exercise we have $(s\sigma)^{\mathbf{I},B} = s^{\mathbf{I},\mathbf{A}}$ for any term s.

Then

$$\begin{aligned}
&[P(s_1, \ldots, s_n)\sigma]^{\mathbf{I},\mathbf{B}} = \mathbf{t} \\
&\text{iff } [P(s_1\sigma, \ldots, s_n\sigma)]^{\mathbf{I},\mathbf{B}} = \mathbf{t} \\
&\text{iff } ((s_1\sigma)^{\mathbf{I},\mathbf{B}}, \ldots, (s_n\sigma)^{\mathbf{I},\mathbf{B}}) \in P^{\mathbf{I}} \\
&\text{iff } (s_1^{\mathbf{I},\mathbf{A}}, \ldots, s_n^{\mathbf{I},\mathbf{A}}) \in P^{\mathbf{I}} \\
&\text{iff } [P(s_1, \ldots, s_n)]^{\mathbf{I},\mathbf{A}} = \mathbf{t}
\end{aligned}$$

STEP: We only consider the case where $X = (\exists y)X_1$. W.l.o.g. we assume $y \neq x$. First we show that $[((\exists y)X_1)\sigma]^{\mathbf{I},\mathbf{B}} = \mathbf{t}$ implies that $[[(\exists y)X_1]^{\mathbf{I},\mathbf{A}} = \mathbf{t}$.

$$\begin{aligned}
&[((\exists y)X_1)\sigma]^{\mathbf{I},\mathbf{B}} = \mathbf{t} \\
&\text{implies } [(\exists y)(X_1\sigma_y)]^{\mathbf{I},\mathbf{B}} = \mathbf{t} \\
&\text{implies } [(\exists y)(X_1\sigma)]^{\mathbf{I},\mathbf{B}} = \mathbf{t} \\
&\text{implies } [(X_1\sigma)]^{\mathbf{I},\mathbf{B}'} = \mathbf{t} \text{ for some } y\text{-variant } \mathbf{B}' \text{ of } \mathbf{B} \\
&\text{implies } [X_1]^{\mathbf{I},\mathbf{A}'} = \mathbf{t} \text{ for some } y\text{-variant } \mathbf{A}' \text{ of } \mathbf{A} \\
&\text{implies } [(\exists y)X_1]^{\mathbf{I},\mathbf{A}} = \mathbf{t}
\end{aligned}$$

Note that by IH the proposition holds for any subexpression of X, in particular it holds for X_1. I.e. for any assignment $\mathbf{A}$, such that $x^{\mathbf{A}} = t^I$ and $\mathbf{B}$ an x-variant of $\mathbf{A}$, we have $[(X_1\sigma)]^{\mathbf{I},\mathbf{B}} = [X_1]^{\mathbf{I},\mathbf{A}}$. Hence, if we set $\mathbf{A}'$ exactly as $\mathbf{A}$ expect that $y^{\mathbf{A}'} := y^{\mathbf{B}'}$, then $x^{\mathbf{A}'} = t^{\mathbf{I}}$ (as $y \neq x$) and $\mathbf{B}'$ is an x-variant of $\mathbf{A}'$. Hence IH is applicable in line 5 above. Furthermore $\mathbf{A}'$ is a y-variant of $\mathbf{A}$.

The direction $[[(\exists y)X_1]^{\mathbf{I},\mathbf{A}} = \mathbf{t}$ implies $[((\exists x)X_1)\sigma]^{\mathbf{I},\mathbf{B}} = \mathbf{t}$ follows similarly. □

7.4 Exercise 5.3.2

7.5 Exercise 5.3.6

Optional Exercises.

1. Exercise 5.3.4

2. Exercise 5.3.5