Exercises.

8.0 Study Chapter 5.4–5.8

8.1 Exercise 5.5.1

Solution. We only consider the claim for γ-formulas, the reasoning for δ-formulas is analog.

Assume $\gamma = (\forall y)A$. Then we consider the substitutions $\sigma = \{y \mapsto x\}$ and $\tau = \{x \mapsto t\}$. Firstly we argue that σ is free for A. This follows as x does not occur in γ and hence cannot occur in A. Secondly, we observe that τ is free for $A\sigma$. This follows as t is ground. Moreover we note that the composition of σ and τ equals $\{y \mapsto t\}$. In summary we obtain

$$\gamma(t) = A\{y \mapsto t\} = (A\{y \mapsto x\})\{x \mapsto t\} = \gamma(x)\{x \mapsto t\} ,$$

where the second equality follows from Theorem 5.2.13.

The case where $\gamma = \neg(\exists y)A$ is proven in exactly the same way. □

8.2 Exercise 5.7.1

Solution. Firstly, we show that the validity of $A\{x \mapsto p\}$ implies the validity of $(\forall x)A$. We have to show that $(\forall x)A$ is true in every model $\mathbf{M} = (\mathbf{D}, \mathbf{I})$. I.e., $[(\forall x)A]^{\mathbf{I},\mathbf{A}} = \mathbf{t}$ for every interpretation $\mathbf{I}$ and assignment $\mathbf{A}$.

Let $\mathbf{I}, \mathbf{A}$ be arbitrary but fixed. We have to show that $A^{\mathbf{I},\mathbf{B}} = \mathbf{t}$ for every x-variant $\mathbf{B}$ of $\mathbf{A}$. We fix an arbitrary x-variant $\mathbf{B}$ of $\mathbf{A}$. If $x^B = d \in \mathbf{D}$, then define an interpretation $\mathbf{I}'$ such that $p^{\mathbf{I}'} = d$. By assumption

$$[A\{x \mapsto p\}]^{\mathbf{I}',\mathbf{B}} = \mathbf{t} .$$

By Prop. 5.3.7 we conclude that

$$A^{\mathbf{I}',\mathbf{B}} = \mathbf{t} .$$

As p is a parameter p does not occur in A, hence the latter equations yields that $A^{\mathbf{I},\mathbf{B}} = \mathbf{t}$, as desired. The interpretation $\mathbf{I}$ and the assignments $\mathbf{A}, \mathbf{B}$ were all arbitrary, hence this is sufficient to show that $(\forall x)A$ is valid.

Secondly, we show that the validity of $(\forall x)A$ implies the validity of $A\{x \mapsto p\}$. We have to show that $A\{x \mapsto p\}$ is true in every model $\mathbf{M} = (\mathbf{D}, \mathbf{I})$.

As above, let $\mathbf{I}, \mathbf{A}$ be arbitrary but fixed. We define an x-variant $\mathbf{B}$ of $\mathbf{A}$ by setting $x^{\mathbf{B}} = p^{\mathbf{I}}$ and conclude from the assumption that $A^{\mathbf{I},\mathbf{B}} = \mathbf{t}$. Now, we observe that the substitution $\sigma := \{x \mapsto p\}$ is free for A. (Here, we use the fact that p is a constant, i.e., a closed term.)

Thus by Prop. 5.3.8 we conclude that

$$[A\{x \mapsto p\}]^{\mathbf{I},\mathbf{A}} = \mathbf{t}$$

The interpretation $\mathbf{I}$ and the assignments $\mathbf{A}$ were arbitrary, hence this is sufficient to show that $A\{x \mapsto p\}$ is valid.

Remark 1. Note that the assumption that p is a parameter is indeed necessary. Consider e.g. the formula A:

$$(\forall y)R(y, y) \to R(p, x) \ .$$

Then clearly $A\{x \mapsto p\}$ is valid, while $(\forall x)A$ is not. To see the latter simple choose a model $\mathbf{M} = (\mathbf{D}, \mathbf{I})$ with at least two elements and an assignment $\mathbf{A}$, such that $x^{\mathbf{A}} \neq p^{\mathbf{I}}$.

□

Optional Exercises.

1. Exercise 5.4.1
2. Exercise 5.4.2
3. Exercise 5.5.2
4. Exercise 5.6.1
5. Exercise 5.6.3
6. Exercise 5.8.2
7. Exercise 5.8.3