Exercises.

9.0 Study Chapter 5

9.1 Exercise 5.3.4

Solution. According to Definition 5.3.6 a formula is true in a model provided its interpretation evaluates to true for all assignments. So it has to be shown that $(X \equiv Y)^{\mathbf{I},\mathbf{A}} = \mathbf{t}$ iff $(X^{\mathbf{I},\mathbf{A}} = Y^{\mathbf{I},\mathbf{A}})$. But this follows from the fact that $(X^{\mathbf{I},\mathbf{A}} \equiv Y^{\mathbf{I},\mathbf{A}}) = \mathbf{t}$ is equivalent as well to the first equation (by Definition 5.3.5) as to the second (by the definition of $\equiv$). $\square$

9.2 Exercise 5.3.5

Solution. First we consider the "if" part: Suppose S is satisfiable in a model for L'. Then all formulas in S are true in some model $\mathbf{M} = (\mathbf{B}, \mathbf{I})$ with assignment $\mathbf{A}$. We define a model $\mathbf{N}$ for L by setting $\mathbf{N} = (\mathbf{D}, \mathbf{I}|_{\mathsf{L}})$, where $\mathbf{I}|_{\mathsf{L}}$ denotes the restriction of the interpretation $\mathbf{I}$ to the language L. Set $\mathbf{J} = \mathbf{I}|_{\mathsf{L}}$. Now it is easy to show by induction that for each $X \in S$, $[X]^{\mathbf{J},\mathbf{A}} = \mathbf{t}$. Hence S is satisfiable in a model for L.

Secondly, consider the "only if" part: Suppose S is satisfiable in a model for L. Then all formulas in S are true in some model $\mathbf{M} = (\mathbf{B}, \mathbf{I})$ with assignment $\mathbf{A}$. We define $\mathbf{N}$ for L' by setting $\mathbf{N} = (\mathbf{J}, \mathbf{D})$, where $\mathbf{J}$ is chosen such that $\mathbf{J}_{\mathsf{L}} = \mathbf{I}$. (The symbols in $\mathsf{L}' - \mathsf{L}$ can be interpreted arbitrarily.) Then it is again easy to show by induction that for each $X \in S$, $[X]^{\mathbf{J},\mathbf{A}} = \mathbf{t}$. Thus S is satisfiable in a model for L'. $\square$

9.3 Exercise 5.4.1

Solution. Let X be a formula of L, we have to show that $[X]^{\mathbf{I},\mathbf{A}} = [X\mathbf{A}]^{\mathbf{I}}$ for all assignments $\mathbf{A}$. To this avail, we exploit Prop. 5.3.7 in its simple form stating that the equality $[X]^{\mathbf{I},\mathbf{A}} = [X\{x \mapsto \mathbf{A}(x)\}]^{\mathbf{I},\mathbf{A}}$ holds. As X is finite the number of variables occurring free in X is finite, too. Suppose $\mathsf{fvar}(X) = \{x_1, \ldots, x_n\}$. Then

$$
\begin{array}{lll}
[X]^{\mathbf{I},\mathbf{A}} = & [X\{x_1 \mapsto \mathbf{A}(x_1)\}]^{\mathbf{I},\mathbf{A}} & \text{Prop. 5.3.7} \\
= & [X\{x_1 \mapsto \mathbf{A}(x_1)\}\{x_2 \mapsto \mathbf{A}(x_2)\}]^{\mathbf{I},\mathbf{A}} & \\
 & \vdots & \\
= & [X\{x_1 \mapsto \mathbf{A}(x_1)\} \cdots \{x_n \mapsto \mathbf{A}(x_n)\}]^{\mathbf{I},\mathbf{A}} & \\
= & [X\mathbf{A}]^{\mathbf{I},\mathbf{A}} & \\
= & [X\mathbf{A}]^{\mathbf{I}} & \text{as } X\mathbf{A} \text{ is closed}
\end{array}
$$

Alternatively a direct proof of Proposition 5.4.3 would procced by induction on X as follows:

a) For the atomic cases this follows from Proposition 5.4.2.

b) Case $X = \neg X_1$: By induction hypothesis (IH) $X_1^{\mathbf{I},\mathbf{A}} = [X_1\mathbf{A}]^{\mathbf{I}}$ which implies $\neg[X_1^{\mathbf{I},\mathbf{A}}] = \neg[[X_1\mathbf{A}]^{\mathbf{I}}]$. The induction step follows as

$$[\neg X_1]^{\mathbf{I},\mathbf{A}} = \neg[X_1^{\mathbf{I},\mathbf{A}}] = \neg[[X_1\mathbf{A}]^{\mathbf{I}}] = [[\neg X_1]\mathbf{A}]^{\mathbf{I}} .$$

c) Case $X = (X_1 \circ X_2)$: Analogous to b).

d) Case $X = (\forall x)X_1$: By IH we know that $[X_1]^{\mathbf{I},\mathbf{B}} = [X_1\mathbf{B}]^{\mathbf{I}}$. Note that we can conceive the assignment $\mathbf{B}$ as a substitution (of closed terms) and thus obtain the equality: $\mathbf{B} = (\mathbf{B}_x)\{x \mapsto B(x)\}$.

Furthermore IH is applicable to the formula $X\mathbf{B}_x$, hence we obtain $[X\mathbf{B}_x]^{\mathbf{I},\mathbf{C}} = [(X\mathbf{B}_x)\mathbf{C}]^{\mathbf{I}}$ for any assignment $\mathbf{C}$. If we assume further that $\mathbf{C}$ is defined such that $\mathbf{C}(x) = \mathbf{B}(x)$, and $\mathbf{C}(y)$ is arbitrary for $x \neq y$, then we obtain $[X\mathbf{B}_x]^{\mathbf{I},\mathbf{C}} = [(X\mathbf{B}]^{\mathbf{I}}$, as $(X\mathbf{B}_x)\mathbf{C} = X\mathbf{B}$.

Now we show that $[(\forall x)X]^{\mathbf{I},\mathbf{A}} = \mathbf{t}$ iff $[(\forall x)X\mathbf{A}]^{\mathbf{I}} = \mathbf{t}$:

$$
\begin{array}{rll}
[(\forall x)X]^{\mathbf{I},\mathbf{A}} = \mathbf{t} & \text{iff} & [X]^{\mathbf{I},\mathbf{B}} = \mathbf{t} \text{ for all } x\text{-variants } \mathbf{B} \text{ of } \mathbf{A} \\
 & \text{iff} & [X\mathbf{B}]^{\mathbf{I}} = \mathbf{t} \text{ for all } x\text{-variants } \mathbf{B} \text{ of } \mathbf{A} \text{ (by IH)} \\
 & \text{iff} & [X\mathbf{B}_x]^{\mathbf{I},\mathbf{C}} = \mathbf{t} \text{ for all assignments } \mathbf{C}, \\
 & & \text{s.t. } \mathbf{C}(x) = \mathbf{B}(x) \text{ and } \mathbf{C} \text{ arbitrary otherwise} \\
 & & \text{and all } x\text{-variants } \mathbf{B} \text{ of } \mathbf{A} \\
 & \text{iff} & [(\forall x)X\mathbf{B}_x]^{\mathbf{I},\mathbf{D}} = \mathbf{t} \text{ for all assignments } \mathbf{D} \\
 & \text{iff} & [(\forall x)X\mathbf{B}_x]^{\mathbf{I}} = \mathbf{t} \text{ } ((\forall x)X\mathbf{B}_x \text{ is closed}) \\
 & \text{iff} & [((\forall x)X)\mathbf{A}]^{\mathbf{I}} = \mathbf{t} \text{ as } \mathbf{B} \text{ is an } x\text{-variant of } \mathbf{A}.
\end{array}
$$

e) Analogous to d).

□

9.4 Exercise 5.4.2

Solution. We only consider the first case. Suppose $\mathbf{M} = (\mathbf{D}, \mathbf{I})$.

$$
\begin{aligned}
(\forall x)X \text{ is true in } \mathbf{M} \quad & \text{iff} \quad [(\forall x)X]^{\mathbf{I},\mathbf{A}} = \mathbf{t} \text{ for all assignments } \mathbf{A} \\
& \text{iff} \quad [X]^{\mathbf{I},\mathbf{B}} = \mathbf{t} \text{ for all } x\text{-variants } \mathbf{B} \text{ of } \mathbf{A} \\
& \text{iff} \quad [(X\{x \mapsto \mathbf{B}(x)\})]^{\mathbf{I},\mathbf{A}} = \mathbf{t} \text{ for all } x\text{-variants } \mathbf{B} \text{ of } \mathbf{A} \\
& \qquad \text{as } \mathbf{B} \text{ is an } x\text{-variant of } \mathbf{A} \text{ by the general statement of Prop. 5.3.7} \\
& \text{iff} \quad [(X\{x \mapsto d\})]^{\mathbf{I},\mathbf{A}} = \mathbf{t} \text{ for all } d \in \mathbf{D}.
\end{aligned}
$$

□