

# Logic LVA 703600 VU3

<http://cl-informatik.uibk.ac.at/teaching/ws05/logic/>

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## Logical Consequence

### Definition

first-order consequence relation

a **sentence**  $X$  is a **logical consequence** of a set  $S$ , if  $X$  is true in every model in which all members of  $S$  are true; denoted  $S \models_f X$

**Note:** definition only for sentences

two generalisations to **formulas** are possible;  $S \models X$  could mean

- ➔ for every  $\mathbf{M}$ , if for all  $Y \in S$ ,  $Y$  is true in  $\mathbf{M}$ , then  $X$  is true in  $\mathbf{M}$ ,

Example:  $P(x) \models (\forall x)P(x)$ ,

or

- ➔ for every  $\mathbf{M}$ , for every  $\mathbf{A}$ , if for all  $Y \in S$ ,  $Y$  is true in  $\mathbf{M}$  under  $\mathbf{A}$ , then  $X$  is true in  $\mathbf{M}$  under  $\mathbf{A}$

Example:  $P(x) \not\models (\forall x)P(x)$

## Compactness Theorem (revisited)

### Theorem

$S \models X$  iff  $S_0 \models X$  for some finite subset  $S_0 \subseteq S$

### Proof

assume  $S \models X$

- ➔ by assumption  $S \cup \{\neg X\}$  is not satisfiable
- ➔ by compactness  $S_0 \cup \{\neg X\}$  is not satisfiable for some  $S_0 \subseteq S$
- ➔ hence  $S_0 \models X$

we have shown:  $S \models X$  implies  $S_0 \models X$  for some finite  $S_0 \subseteq S$   $\square$

## First-Order Semantic Tableaux

let  $\{A_1, \dots, A_n\}$  be a set of sentences

- ➔ the one-branch tree

$$\begin{array}{c} A_1 \\ \vdots \\ A_n \end{array}$$

is a **tableau** (for  $\{A_1, \dots, A_n\}$ )

- ➔ tableau expansion rules

$$\frac{\neg\neg Z}{Z}$$

$$\frac{\neg\top}{\perp}$$

$$\frac{\neg\perp}{\top}$$

$$\frac{\alpha}{\alpha_1}$$

$$\frac{\beta}{\beta_1|\beta_2}$$

$$\alpha_2$$

$$\frac{\gamma}{\gamma(t)}$$

for any  $t \in L^{\text{par}}$

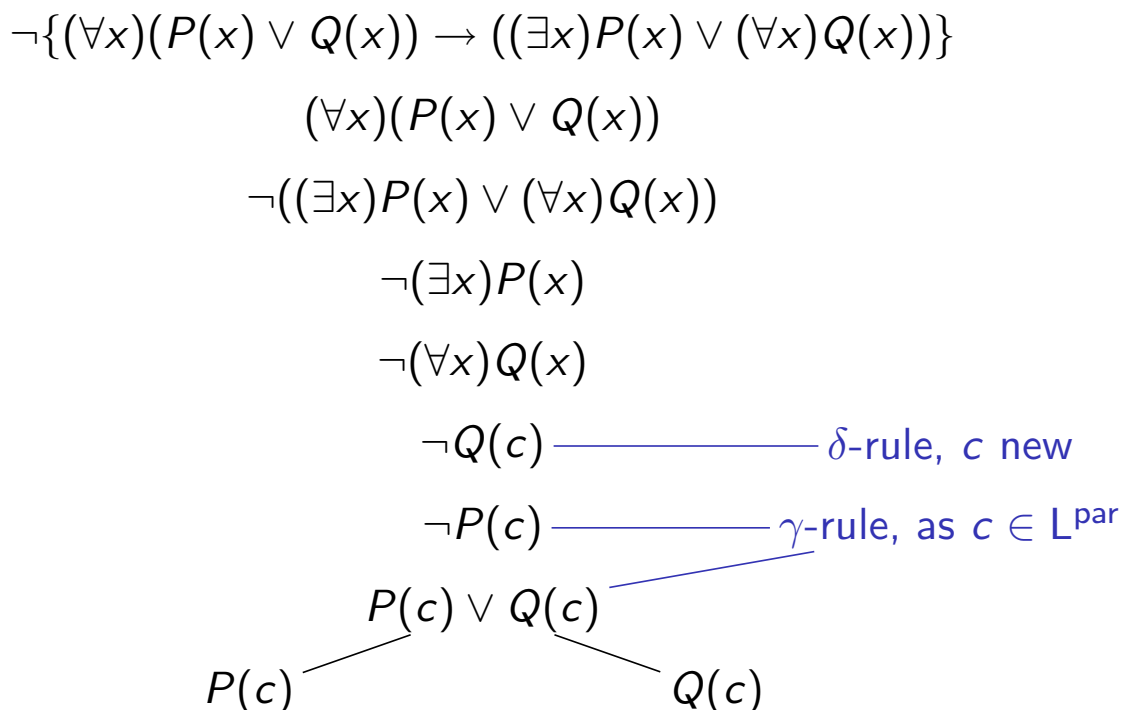
$$\frac{\delta}{\delta(p)}$$

for some new parameter  $p$

- ➔ suppose  $\mathbf{T}$  is a tableau for  $\{A_1, \dots, A_n\}$ ,  $\mathbf{T}^*$  obtained by applying a **tableau expansion rule** to  $\mathbf{T}$ , then  $\mathbf{T}^*$  is a **tableau**

### Example

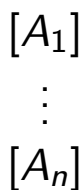
proof of  $(\forall x)(P(x) \vee Q(x)) \rightarrow ((\exists x)P(x) \vee (\forall x)Q(x))$ :



### Resolution

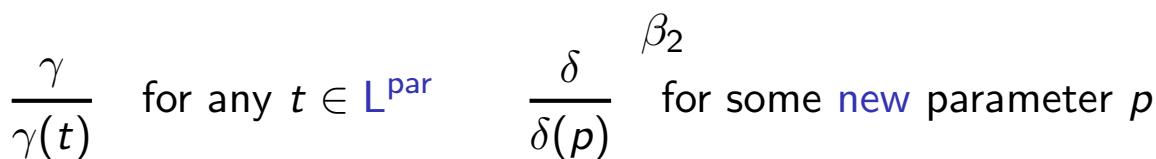
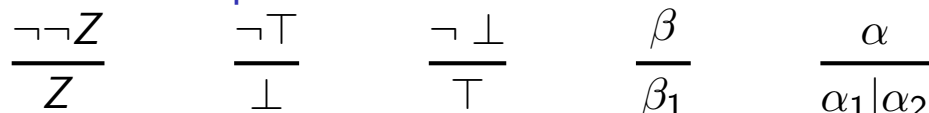
let  $\{A_1, \dots, A_n\}$  be a set of sentences

➔ the sequence of disjunctions



is a **resolution expansion** (for  $\{A_1, \dots, A_n\}$ )

➔ resolution expansion rules



➔ suppose **R** is a resolution expansion and **R\*** obtained by applying an **expansion or resolution rule** to **R**, then **R\*** is a **resolution expansion**

## Undecidability of First-Order Logic

### Theorem

halting problem for TMs is **undecidable**

we set  $HP = \{\ulcorner M \urcorner \# \ulcorner x \urcorner \mid \text{TM } M \text{ halts on input } x\}$

### Theorem

the validity problem for first-order formulas is **undecidable**

### Proof

(sketch)

- ➔ reduction from halting problem
- ➔ define  $\sigma: HP \rightarrow \text{FOL}$  s.t.  $\ulcorner M \urcorner \# \ulcorner x \urcorner \in HP$  iff first-order formalisation of “TM  $M$  halts on input  $x$ ” is valid □

### Theorem

no refinement of semantic tableaux (or resolution) can exist, whose termination can be guaranteed

the notion of **satisfiability** extends to proof-structures

- ➔ a tableau **branch** is **satisfiable** if the set of sentences on it is satisfiable
- ➔ a **tableau** is **satisfiable** if some branch is satisfiable
- ➔ a **resolution expansion** is **satisfiable** if in some model, every disjunction in the expansion is true

### Lemma

- ➔ tableau expansion rules preserve satisfiability
- ➔ resolution expansion rules and the resolution rule preserve satisfiability

**Proof**

suppose  $\mathbf{T}$  is a satisfiable tableau and an expansion rule is applied to  $\mathbf{T}$  to obtain  $\mathbf{T}^*$

proof by **case-distinction** on the rules applied, we only consider the case of a  $\gamma$ -rule

$$\frac{\gamma}{\gamma(t)}$$

we use the proposition

- ➔ if  $S \cup \{\gamma\}$  is satisfiable, so is  $S \cup \{\gamma, \gamma(t)\}$  for any closed term  $t$

now

- ➔ suppose the branch  $\tau \in \mathbf{T}$  is satisfiable over  $L^{\text{par}}$
- ➔ we can assume that  $S \cup \{\gamma\}$  denotes the set of sentences on  $\tau$
- ➔ by **above proposition**:  $S \cup \{\gamma\} \cup \{\gamma(t)\}$  is **satisfiable** (over  $L^{\text{par}}$ )

□

**Soundness****Theorem****Soundness Theorem**

- ➔ If  $X$  has a tableau proof, then  $X$  is valid.
- ➔ If  $X$  has a resolution proof, then  $X$  is valid.

to show completeness we define

- ➔ a finite set of sentences of  $L^{\text{par}}$  is **tableau consistent** if there is no closed tableau for it
- ➔  $\mathcal{C}$  = the collection of all tableau-consistent sets

### Lemma

$\mathcal{C}$  is a first-order consistency property

### Proof

- ➔ the proof is by **case-distinction** on the definition of FCP
- ➔ let  $S \in \mathcal{C}$
- ➔ we consider the case  $\beta \in S$  and have to show that either  $S \cup \{\beta_1\} \in \mathcal{C}$  or  $S \cup \{\beta_2\} \in \mathcal{C}$

**proof by contradiction:** assume

1.  $S \cup \{\beta_1\} \notin \mathcal{C}$
2.  $S \cup \{\beta_2\} \notin \mathcal{C}$

by 1)+2) there are closed tableaux  $\mathbf{T}_1, \mathbf{T}_2$  for  $S \cup \{\beta_1\}, S \cup \{\beta_2\}$

make  $\mathbf{T}_1, \mathbf{T}_2$  compatible by **renaming parameters**

combine  $\mathbf{T}_1, \mathbf{T}_2$  to a closed tableau for  $S$

contradiction

□

## Completeness

### Theorem

#### Completeness of First-Order Tableaux

If the sentence  $X$  of  $L$  is valid,  $X$  has a tableau proof.

### Proof

**proof by contradiction:** assume there is no closed tableau for  $\neg X$

- ➔ hence  $\{\neg X\}$  is in  $\mathcal{C}$
- ➔ hence  $\{\neg X\}$  is satisfiable, contradiction to  $X$  is valid

□

### Theorem

#### Completeness of First-Order Resolution

If the sentence  $X$  of  $L$  is valid,  $X$  has a resolution proof.

## Summary

- ➔ consequence relation  $\models$
- ➔ first-order semantic tableaux
- ➔ first-order resolution
- ➔ undecidability of the validity problem
- ➔ soundness of tableaux & resolution
- ➔ completeness of tableaux & resolution