Logic LVA 703600 VU3

http://cl-informatik.uibk.ac.at/teaching/ws05/logic/

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Logical Consequence

Definition first-order consequence relation

a sentence X is a logical consequence of a set S, if X is true in every model in which all members of S are true; denoted $S \models_f X$

Note: definition only for sentences

two generalisations to formulas are possible; $S \models X$ could mean

- for every $\mathbf{M}$, if for all $Y \in S$, Y is true in $\mathbf{M}$, then X is true in $\mathbf{M}$,

  Example: $P(x) \models (\forall x)P(x)$,

  or
- for every $\mathbf{M}$, for every $\mathbf{A}$, if for all $Y \in S$, Y is true in $\mathbf{M}$ under $\mathbf{A}$, then X is true in $\mathbf{M}$ under $\mathbf{A}$

  Example: $P(x) \not\models (\forall x)P(x)$

Compactness Theorem (revisited)

Theorem $S \models X$ iff $S_0 \models X$ for some finite subset $S_0 \subseteq S$

Proof assume $S \models X$

- by assumption $S \cup \{\neg X\}$ is not satisfiable
- by compactness $S_0 \cup \{\neg X\}$ is not satisfiable for some $S_0 \subseteq S$
- hence $S_0 \models X$

we have shown: $S \models X$ implies $S_0 \models X$ for some finite $S_0 \subseteq S$ $\square$

First-Order Semantic Tableaux

let $\{A_1, \dots, A_n\}$ be a set of sentences

- the one-branch tree

$$\begin{array}{c} A_1 \\ \vdots \\ A_n \end{array}$$

  is a tableau (for $\{A_1, \dots, A_n\}$)
- tableau expansion rules

$$\frac{\neg\neg Z}{Z} \qquad \frac{\neg\top}{\bot} \qquad \frac{\neg\bot}{\top} \qquad \frac{\alpha}{\begin{array}{c}\alpha_1\\ \alpha_2\end{array}} \qquad \frac{\beta}{\beta_1 | \beta_2}$$

$$\frac{\gamma}{\gamma(t)} \text{ for any } t \in \mathsf{L}^{\mathsf{par}} \qquad \frac{\delta}{\delta(p)} \text{ for some new parameter } p$$

- suppose $\mathbf{T}$ is a tableau for $\{A_1, \dots, A_n\}$, $\mathbf{T}^*$ obtained by applying a tableau expansion rule to $\mathbf{T}$, then $\mathbf{T}^*$ is a tableau

Example

proof of $(\forall x)(P(x) \lor Q(x)) \to ((\exists x)P(x) \lor (\forall x)Q(x))$:

$$\neg\{(\forall x)(P(x) \lor Q(x)) \to ((\exists x)P(x) \lor (\forall x)Q(x))\}$$
$$(\forall x)(P(x) \lor Q(x))$$
$$\neg((\exists x)P(x) \lor (\forall x)Q(x))$$
$$\neg(\exists x)P(x)$$
$$\neg(\forall x)Q(x)$$
$\neg Q(c)$ — δ-rule, c new

$\neg P(c)$ — γ-rule, as $c \in \mathsf{L}^{par}$

$P(c) \lor Q(c)$

$P(c)$ | $Q(c)$

Resolution

let $\{A_1, \dots, A_n\}$ be a set of sentences

- ➡ the sequence of disjunctions

$$[A_1]$$
$$\vdots$$
$$[A_n]$$

is a resolution expansion (for $\{A_1, \dots, A_n\}$)

- ➡ resolution expansion rules

$$\frac{\neg\neg Z}{Z} \qquad \frac{\neg\top}{\bot} \qquad \frac{\neg\bot}{\top} \qquad \frac{\beta}{\begin{matrix}\beta_1\\ \beta_2\end{matrix}} \qquad \frac{\alpha}{\alpha_1 | \alpha_2}$$

$$\frac{\gamma}{\gamma(t)} \text{ for any } t \in \mathsf{L}^{par} \qquad \frac{\delta}{\delta(p)} \text{ for some new parameter } p$$

- ➡ suppose $\mathbf{R}$ is a resolution expansion and $\mathbf{R}^*$ obtained by applying an expansion or resolution rule to $\mathbf{R}$, then $\mathbf{R}^*$ is a resolution expansion

Undecidability of First-Order Logic

Theorem halting problem for TMs is undecidable

we set HP $= \{\ulcorner M \urcorner \# \ulcorner x \urcorner \mid \text{TM } M \text{ halts on input } x\}$

Theorem the validity problem for first-order formulas is undecidable

Proof (sketch)

- ➡ reduction from halting problem
- ➡ define $\sigma\colon \text{HP} \to \text{FOL}$ s.t. $\ulcorner M \urcorner \# \ulcorner x \urcorner \in \text{HP}$ iff first-order formalisation of "TM M halts on input x" is valid □

Theorem no refinement of semantic tableaux (or resolution) can exist, whose termination can be guaranteed

the notion of satisfiability extends to proof-structures

- ➡ a tableau branch is satisfiable if the set of sentences on it is satisfiable
- ➡ a tableau is satisfiable if some branch is satisfiable
- ➡ a resolution expansion is satisfiable if in some model, every disjunction in the expansion is true

Lemma

- ➡ tableau expansion rules preserve satisfiability
- ➡ resolution expansion rules and the resolution rule preserve satisfiability

Proof

suppose **T** is a satisfiable tableau and an expansion rule is applied to **T** to obtain $\mathbf{T}^*$

proof by case-distinction on the rules applied, we only consider the case of a γ-rule

$$\frac{\gamma}{\gamma(t)}$$

we use the proposition

- if $S \cup \{\gamma\}$ is satisfiable, so is $S \cup \{\gamma, \gamma(t)\}$ for any closed term t

now

- suppose the branch $\tau \in \mathbf{T}$ is satisfiable over $\mathsf{L}^{\mathsf{par}}$
- we can assume that $S \cup \{\gamma\}$ denotes the set of sentences on τ
- by above proposition: $S \cup \{\gamma\} \cup \{\gamma(t)\}$ is satisfiable (over $\mathsf{L}^{\mathsf{par}}$)

□

Soundness

Theorem Soundness Theorem

- If X has a tableau proof, then X is valid.
- If X has a resolution proof, then X is valid.

to show completeness we define

- a finite set of sentences of $\mathsf{L}^{\mathsf{par}}$ is tableau consistent if there is no closed tableau for it
- $\mathcal{C}$ = the collection of all tableau-consistent sets

Lemma $\mathcal{C}$ is a first-order consistency property

Proof

- the proof is by case-distinction on the definition of FCP
- let $S \in \mathcal{C}$
- we consider the case $\beta \in S$ and have to show that either $S \cup \{\beta_1\} \in \mathcal{C}$ or $S \cup \{\beta_2\} \in \mathcal{C}$

  proof by contradiction: assume
  1. $S \cup \{\beta_1\} \notin \mathcal{C}$
  2. $S \cup \{\beta_2\} \notin \mathcal{C}$

  by 1)+2) there are closed tableaux $\mathbf{T}_1$, $\mathbf{T}_2$ for $S \cup \{\beta_1\}$, $S \cup \{\beta_2\}$

  make $\mathbf{T}_1$, $\mathbf{T}_2$ compatible by renaming parameters

  combine $\mathbf{T}_1$, $\mathbf{T}_2$ to a closed tableau for S

  contradiction

□

Completeness

Theorem Completeness of First-Order Tableaux

If the sentence X of L is valid, X has a tableau proof.

Proof

proof by contradiction: assume there is no closed tableau for $\neg X$

- hence $\{\neg X\}$ is in $\mathcal{C}$
- hence $\{\neg X\}$ is satisfiable, contradiction to X is valid

□

Theorem Completeness of First-Order Resolution

If the sentence X of L is valid, X has a resolution proof.

Summary

- consequence relation $\models$
- first-order semantic tableaux
- first-order resolution
- undecidability of the validity problem
- soundness of tableaux & resolution
- completeness of tableaux & resolution