Logic LVA 703600 VU3

http://cl-informatik.uibk.ac.at/teaching/ws05/logic/

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Propositional Logic

"elementary sentences" atomic formulas

"Mountains are high."	M
"Snow is white."	S

form complex sentences by using connectives

"Mountains are high and snow is white."	$M \wedge S$
"If mountains are high, then snow is white."	$M \rightarrow S$

- ➡ How does the truth/falsity of complex sentences depends on the truth/falsity of their atoms?
- ➡ Which are true, independent of the status of their atoms?

Syntax of Propositional Logic

infinite set of propositional letters:

$$P_1, P_2, \ldots$$

connectives

$\top$, $\bot$	arity 0	constants
$\neg$	arity 1	
$\wedge$, $\vee$, etc.	arity 2	
ite	arity 3	
	$\vdots$	

auxiliary symbols: (and)

Definition

atomic formula

propositional letter, $\top, \bot$

Definition

the set of propositional formulas **P** is the smallest set, such that

- ➡ if A is an atomic formula, then $A \in \mathbf{P}$
- ➡ $X \in \mathbf{P}$ implies $\neg X \in \mathbf{P}$
- ➡ $X, Y \in \mathbf{P}$ implies $(X \circ Y) \in \mathbf{P}$ $\quad$ $\circ$ binary

alternatively

Definition

Base: if A is an atomic formula, then $A \in \mathbf{P}$

Step: suppose $X, Y \in \mathbf{P}$

- ➡ $\neg X \in \mathbf{P}$ and
- ➡ $(X \circ Y) \in \mathbf{P}$

definitions are equivalent; proof by structural induction

Theorem principle of structural induction

every formula in **P** has property **Q** if

Base: every atomic formula has property **Q**

Step:
- if X has property **Q**, so does $\neg X$
- if X and Y have property **Q**, so does $(X \circ Y)$

Theorem principle of structural recursion

there is exactly one function f defined on **P** such that

Base: value of f is specified for atoms

Step:
- value of f on $\neg X$ is specified in terms of value of f on X
- value of f on $(X \circ Y)$ is specified in terms of value of f on X and Y

Unique Parsing

Theorem

every propositional formula is in exactly one of the following categories:

- atomic
- $\neg X$, for unique X
- $(X \circ Y)$, for unique $\circ$ and unique X and Y

Proof

- show that any formula in **P** falls in one of the categories; Exercise 2.2.3
- show uniqueness: Exercise 2.2.6 and 2.2.8

□

Subformulas

Definition

- immediate subformulas
  - an atomic formula has no immediate subformulas
  - X is the only immediate subformula of $\neg X$
  - X and Y are the only immediate subformulas of $(X \circ Y)$
- sets of subformulas

  the set of subformulas of X is the smallest set containing X and with each member, the immediate subformulas of that member
- X is called an improper subformula of itself.

Optional Exercise

transform the definition of subformulas into a recursive one.

Semantics of Propositional Logic

$\mathbf{Tr} := \{\mathbf{t}, \mathbf{f}\}$; interpretation of $\neg$ as a mapping $\neg \colon \mathbf{Tr} \to \mathbf{Tr}$:

$$\neg(\mathbf{t}) := \mathbf{f} \qquad \neg(\mathbf{f}) := \mathbf{t}$$

binary connectives

		Primary								Secondary	
		$\wedge$	$\vee$	$\to$	$\leftarrow$	$\uparrow$	$\downarrow$	$\not\to$	$\not\leftarrow$	$\equiv$	$\not\equiv$
t	t	t	t	t	t	f	f	f	f	t	f
t	f	f	t	f	t	t	f	t	f	f	t
f	t	f	t	t	f	t	f	f	t	f	t
f	f	f	f	t	t	t	t	f	f	t	f

ternary connectives

$$\mathrm{ite}(X, Y, Z) \equiv ((X \wedge Y) \vee (\neg X \wedge Z))$$

Definition a Boolean valuation is a mapping $v\colon \mathbf{P} \to \mathbf{Tr}$ fulfilling

- $v(\top) = \mathbf{t}$ $\quad v(\bot) = \mathbf{f}$
- $v(\neg X) = \neg v(X)$
- $v(X \circ Y) = v(X) \circ v(Y)$, $\circ$ binary.

Theorem for each mapping f

$$f\colon \text{propositional letters} \to \mathbf{Tr}$$

exists a Boolean valuation v with $f(P_i) = v(P_i)$

Theorem v_1, v_2 are valuations with $v_1(P) = v_2(P)$, for all $P \in S$, then

$$v_1(X) = v_2(X) \qquad \text{if } X \text{ contains only letters from } S$$

Syntax Structural Induction Unique Parsing Semantics Valuations Tautology Summary

Boolean valuations are uniquely determined by the valuations of the occurring atoms

Example consider $v\colon \mathbf{P} \to \mathbf{Tr}$ such that

$$v(P) = \mathbf{t} \quad v(Q) = \mathbf{f} \quad v(R) = \mathbf{f} \qquad v(X) = \mathbf{f} \text{ if } X \notin \{P, Q, R\}$$

Hence

$$\begin{aligned} v((P \uparrow \neg Q) \to R) &= v(P \uparrow \neg Q) \to v(R) \\ &= (v(P) \uparrow v(\neg Q)) \to v(R) \\ &= (v(P) \uparrow \neg v(Q)) \to v(R) \\ &= (\mathbf{t} \uparrow \neg \mathbf{f}) \to \mathbf{f} \\ &= (\mathbf{t} \uparrow \mathbf{t}) \to \mathbf{f} \\ &= \mathbf{f} \to \mathbf{f} = \mathbf{t} \end{aligned}$$

Convention: outermost brackets are dropped.

Definition

X is a tautology if $v(X) = \mathbf{t}$ for every v

Definition

decidable

P is a decision procedure for W if, for every input

P stops with output *yes* $x \in W$
stops with output *no* $x \notin W$

if there is a decision procedure: W decidable

Theorem

the set of tautologies is decidable

Proof

- Boolean valuations are uniquely determined by the valuations of the occurring atoms
- suppose X contains n letters, 2^n valuations exist
- test with truth-tables

□

Definition

a set S is satisfiable, if there exists some v, such that for all $X \in S$, $v(X) = \mathbf{t}$

Theorem

X is a tautology iff $\{\neg X\}$ is not satisfiable.

Definition

- we say $\bullet$ is the dual of $\circ$ if

$$\text{for } x, y \text{ ranging over } \mathbf{Tr}: \quad \neg(x \circ y) = (\neg x \bullet \neg y)$$

  example: $\vee$ is dual to $\wedge$
- obtain X^d by
  1. replacing every $\top$ in X with $\bot$
  2. replacing every binary symbol in X by its dual
- X^d is called the dual formula of X

Summary

- Syntax of Propositional Logic
- Unique Parsing Theorem
- Semantics of Propositional Logic
- Boolean Valuations
- Propositional Logic is Decidable