Logic   LVA 703600 VU3

http://cl-informatik.uibk.ac.at/teaching/ws05/logic/

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Proof Procedures

How-to prove that a given propositional formula is satisfiable or even a tautology?

truth-tables	natural deduction
normal forms	Hilbert systems
binary decision diagrams	semantic tableaux
Davis-Putnam-Loveland	resolution
⋮	⋮

How-to prove that a given propositional formula is a tautology automatically?

binary decision diagrams	semantic tableaux
Davis-Putnam-Loveland	resolution

semantic tableaux

➡ refutational method
　　➡ to prove X, derive contradiction from $\neg X$
➡ connected to disjunctive normal form (DNF)
➡ extends to first-order logic (and non-classical logics)

resolution

➡ refutational method

➡ connected to conjunctive normal form (CNF)

➡ extends to first-order logic (and non-classical logics)

NB: binary decision diagrams do not extend to first-order, Davis-Putnam-Loveland does, but poorly

Definition   semantic tableau

let $\{A_1, \ldots, A_n\}$ be a set of formulas
➡ the one-branch tree

$$A_1$$
$$A_2$$
$$\vdots$$
$$A_n$$

　is a tableau for $\{A_1, \ldots, A_n\}$
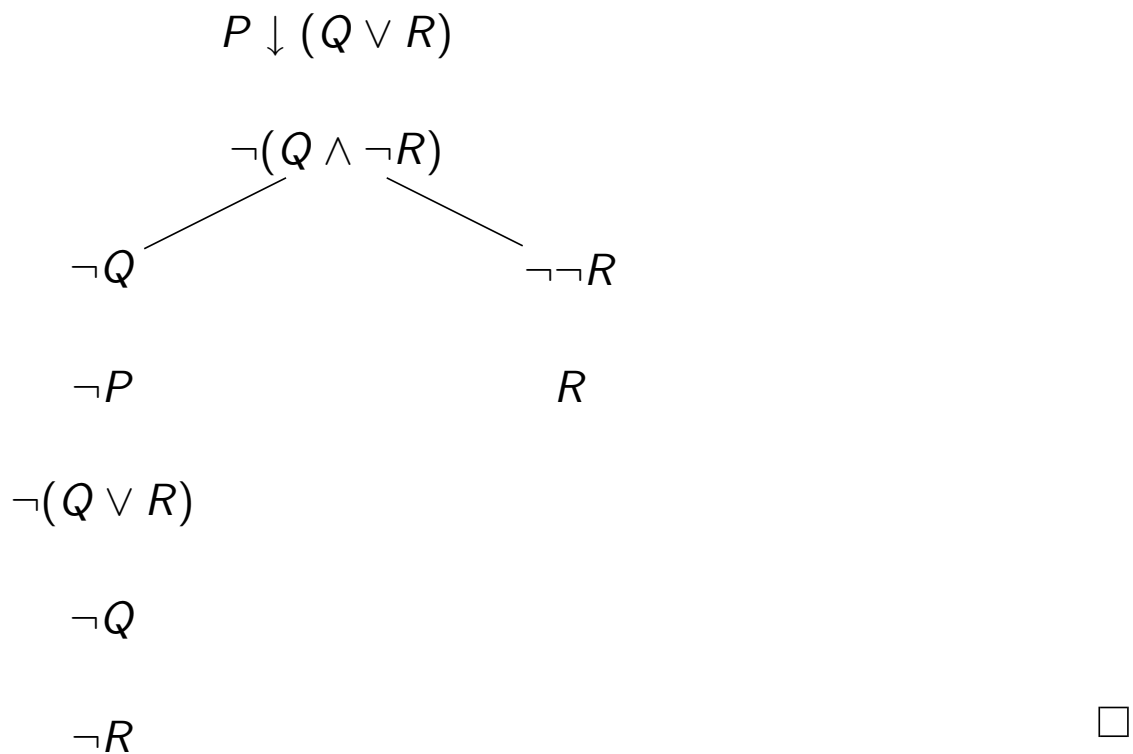
➡ tableau expansion rules

$$\frac{\neg\neg Z}{Z} \qquad \frac{\neg\top}{\bot} \qquad \frac{\neg\bot}{\top} \qquad \frac{\alpha}{\begin{array}{c}\alpha_1\\\alpha_2\end{array}} \qquad \frac{\beta}{\beta_1|\beta_2}$$

　e.g. if $\neg\neg Z$ on a branch, then extend branch with Z

➡ suppose **T** is a tableau for $\{A_1, \ldots, A_n\}$, **T*** obtained by applying a tableau expansion rule to **T**, then **T*** is a tableau for $\{A_1, \ldots, A_n\}$

Example

tableau for $\{P \downarrow (Q \vee R), \neg(Q \wedge \neg R)\}$

$$P \downarrow (Q \vee R)$$

$$\neg(Q \wedge \neg R)$$

$$\neg Q \qquad\qquad\qquad \neg\neg R$$

$$\neg P \qquad\qquad\qquad R$$

$$\neg(Q \vee R)$$

$$\neg Q$$

$$\neg R \qquad\qquad\qquad\qquad\qquad\qquad \square$$

Example

tableau proof for
$$(P \rightarrow (Q \rightarrow R)) \rightarrow ((P \vee S) \rightarrow ((Q \rightarrow R) \vee S))$$

$$\neg((P \rightarrow (Q \rightarrow R)) \rightarrow ((P \vee S) \rightarrow ((Q \rightarrow R) \vee S)))$$
$$(P \rightarrow (Q \rightarrow R))$$
$$\neg((P \vee S) \rightarrow ((Q \rightarrow R) \vee S))$$
$$P \vee S$$
$$\neg((Q \rightarrow R) \vee S)$$
$$\neg(Q \rightarrow R)$$
$$\neg S$$
$$\neg P \qquad\qquad Q \rightarrow R$$
$$P \qquad S$$
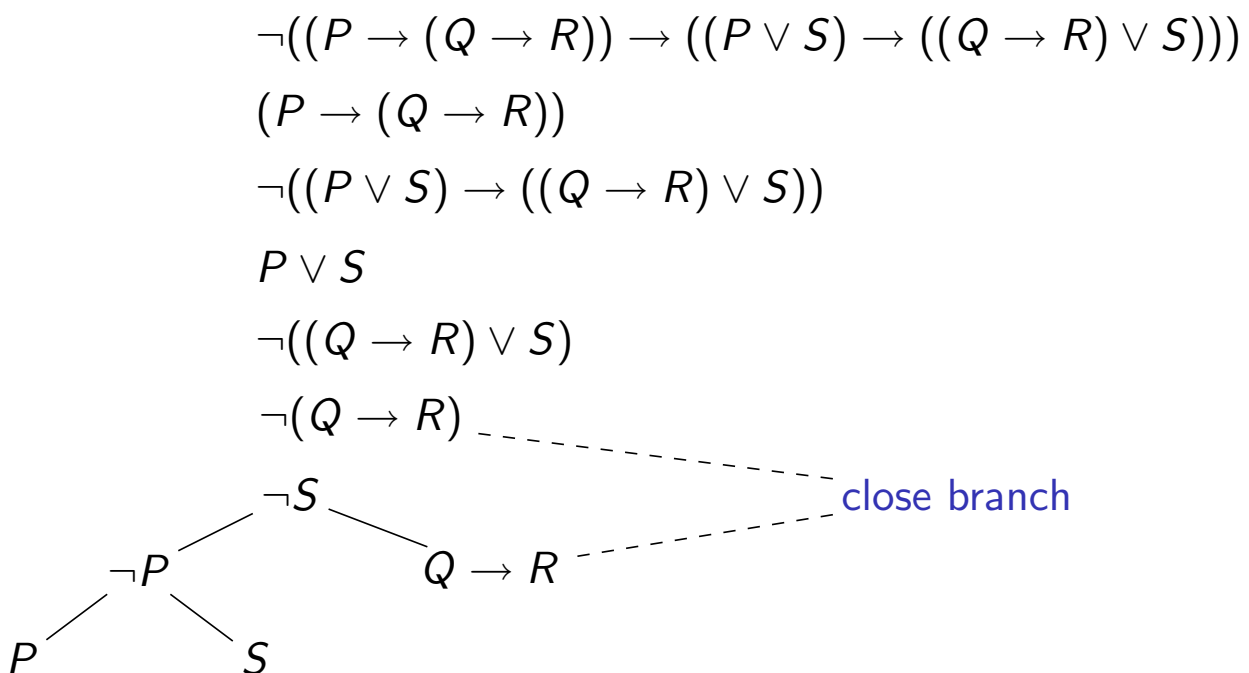
close branch

Definition ▸ tableau proof

➡ a branch is closed if X and $\neg X$, or if $\bot$ occur(s) on it

➡ a tableau is closed if every branch is closed

➡ a tableau proof of X is a closed tableau for $\{\neg X\}$

➡ in a strict tableau no formula is expanded twice on the same branch

Theorem ▸ soundness of propositional tableau

If X has a tableau proof, then X is a tautology.

Definition

➡ a branch is satisfiable if the set of formulas on the branch is satisfiable

➡ a tableau is satisfiable if at least one branch is satisfiable

Theorem ▸ Expansion rules preserve satisfiability.

Proof ▸ suppose $\mathbf{T}$ is satisfiable, with satisfiable branch τ; $\mathbf{T}^*$ is obtained by expansion on branch Θ; we show: $\mathbf{T}^*$ is satisfiable

➡ Case $\tau \neq \Theta$: $\mathbf{T}^*$ is satisfiable as τ is a branch of it

➡ Case $\tau = \Theta$: assume expansion was applied to X; we know: $v(X) = \mathbf{t}$

  ➡ Subcase $X = \alpha$: Prop. 2.6.1 yields $v(X) = v(\alpha_1) \wedge v(\alpha_2)$; hence $v(\alpha_1) = v(\alpha_2) = \mathbf{t}$; extension of Θ is satisfiable

  ➡ Subcase $X = \beta$: Prop. 2.6.1 yields $v(X) = v(\beta_1) \vee v(\beta_2)$; hence $v(\beta_1) = \mathbf{t}$ or $v(\beta_2) = \mathbf{t}$; one of the extensions of Θ is satisfiable

  ➡ Subcases (i) $X = \neg\neg Z$, (ii) $X = \neg\top$, (iii) $X = \neg\bot$: use $v(\neg\neg Z) = v(Z)$, $v(\neg\top) = v(\bot)$ and $v(\neg\bot) = v(\top)$ □

Theorem

If there is a closed tableau for S, then S is not satisfiable.

Proof

suppose S is satisfiable, a closed tableau exists

➡ S is satisfiable, hence the initial one-branch tableau is satisfiable

➡ a closed tableau exists, obtained from the inital tableau

➡ thus the closed tableau is satisfiable   previous theorem

➡ contradiction   definition of satisfiability   □

Proof

of soundness theorem

➡ a tableau proof of X is a closed tableau for $\{\neg X\}$

➡ $\{\neg X\}$ is not satisfiable

➡ no counter-example exists: X is a tautology   □

Definition

resolution expansion rules

$$\frac{\neg\neg Z}{Z} \qquad \frac{\neg\top}{\bot} \qquad \frac{\neg\bot}{\top} \qquad \frac{\beta}{\begin{array}{c}\beta_1\\\beta_2\end{array}} \qquad \frac{\alpha}{\alpha_1 | \alpha_2}$$

Definition

resolvent

D_1, D_2 denote disjunctions with $X, \neg X$ occurring as member; the resolvent D of D_1 and D_2 is obtained by

➡ deleting all occurrences of X from D_1

➡ deleting all occurrences of $\neg X$ from D_2

➡ combine the resulting disjunctions

F is called trivial resolvent of D, if obtained by deleting all occurrences of $\bot$ in D

D follows from D_1, D_2 by the resolution rule if D is the (trivial) resolvent of D_1, D_2 (D')

Definition   resolution

let $\{A_1, \ldots, A_n\}$ be a set of formulas

➡ the sequence of disjunctions

$$[A_1]$$
$$[A_2]$$
$$\vdots$$
$$[A_n]$$

is a resolution expansion for $\{A_1, \ldots, A_n\}$

➡ suppose **R** is a resolution expansion for $\{A_1, \ldots, A_n\}$ and **R*** obtained by applying an expansion or resolution rule to **R**, then **R*** is a resolution expansion for $\{A_1, \ldots, A_n\}$

Definition   resolution proof

➡ a resolution expansion containing the empty clause, is closed

➡ a resolution proof of X is a closed resolution expansion for $\{\neg X\}$

Example   resolution proof of $P \to (Q \to P)$

$$[\neg(P \to (Q \to P))] \quad [P] \quad [\neg(Q \to P)] \quad [Q] \quad [\neg P] \quad []$$

NB: there is no analog of strictness for the resolution rule, but a sequence of expansion rules is strict, if every disjunction is only expanded once

Theorem   soundness of propostional resolution

If X has a resolution proof, then X is a tautology.

Summary

➡ proof procedures

➡ propositional semantic tableaux

➡ tableau implementations

➡ propositional tableau soundness

➡ propositional resolution

➡ propositional resolution soundness