Logic LVA 703600 VU3

http://cl-informatik.uibk.ac.at/teaching/ws05/logic/

Georg Moser (VU)[1] Christian Vogt (VU)[2]

[1]georg.moser@uibk.ac.at
office hours: Thursday 1pm–3pm

[2]christian.vogt@uibk.ac.at
office hours: Tuesday 9am–11am

Autumn 2005

Proof Procedures

How-to prove that a given propositional formula is satisfiable or even a tautology?

truth-tables	natural deduction
normal forms	Hilbert systems
binary decision diagrams	semantic tableaux
Davis-Putnam-Loveland	resolution
⋮	⋮

How-to prove that a given propositional formula is a tautology automatically?

binary decision diagrams	semantic tableaux
Davis-Putnam-Loveland	resolution

semantic tableaux

- refutational method
  - to prove X, derive contradiction from $\neg X$
- connected to disjunctive normal form (DNF)
- extends to first-order logic (and non-classical logics)

resolution

- refutational method
- connected to conjunctive normal form (CNF)
- extends to first-order logic (and non-classical logics)

NB: binary decision diagrams do not extend to first-order, Davis-Putnam-Loveland does, but poorly

Definition semantic tableau

let $\{A_1, \dots, A_n\}$ be a set of formulas

- the one-branch tree

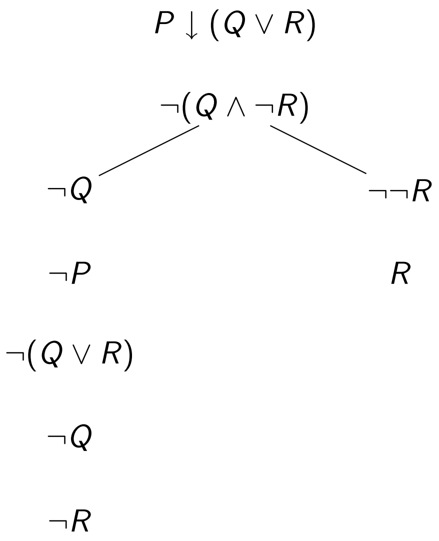
$$\begin{array}{c} A_1 \\ A_2 \\ \vdots \\ A_n \end{array}$$

  is a tableau for $\{A_1, \dots, A_n\}$
- tableau expansion rules

$$\frac{\neg\neg Z}{Z} \qquad \frac{\neg\top}{\bot} \qquad \frac{\neg\bot}{\top} \qquad \frac{\alpha}{\begin{array}{c}\alpha_1 \\ \alpha_2\end{array}} \qquad \frac{\beta}{\beta_1 | \beta_2}$$

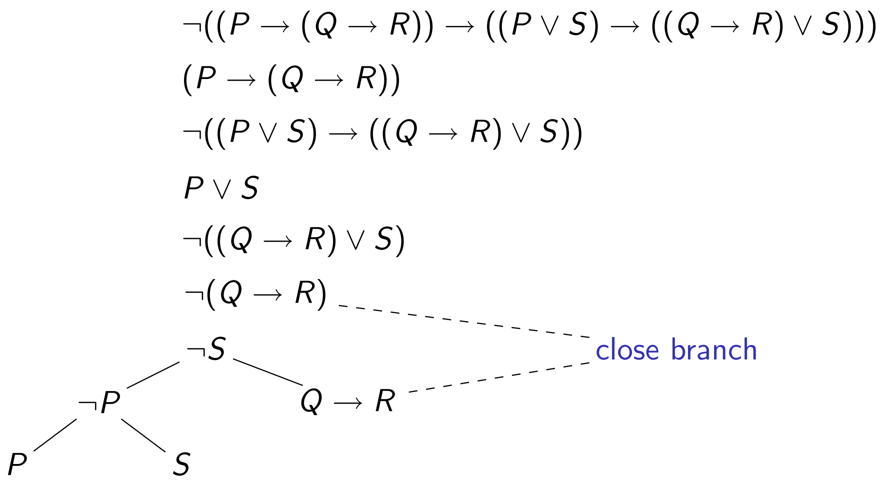
  e.g. if $\neg\neg Z$ on a branch, then extend branch with Z
- suppose $\mathbf{T}$ is a tableau for $\{A_1, \dots, A_n\}$, $\mathbf{T}^*$ obtained by applying a tableau expansion rule to $\mathbf{T}$, then $\mathbf{T}^*$ is a tableau for $\{A_1, \dots, A_n\}$

Example tableau for $\{P \downarrow (Q \vee R), \neg(Q \wedge \neg R)\}$



□

Example tableau proof for
$(P \to (Q \to R)) \to ((P \vee S) \to ((Q \to R) \vee S))$



Definition tableau proof

- a branch is closed if X and $\neg X$, or if $\perp$ occur(s) on it
- a tableau is closed if every branch is closed
- a tableau proof of X is a closed tableau for $\{\neg X\}$
- in a strict tableau no formula is expanded twice on the same branch

Theorem soundness of propositional tableau

If X has a tableau proof, then X is a tautology.

Definition

- a branch is satisfiable if the set of formulas on the branch is satisfiable
- a tableau is satisfiable if at least one branch is satisfiable

Theorem Expansion rules preserve satisfiability.

Proof suppose $\mathbf{T}$ is satisfiable, with satisfiable branch τ; $\mathbf{T}^*$ is obtained by expansion on branch Θ; we show: $\mathbf{T}^*$ is satisfiable

- Case $\tau \neq \Theta$: $\mathbf{T}^*$ is satisfiable as τ is a branch of it
- Case $\tau = \Theta$: assume expansion was applied to X; we know: $v(X) = \mathbf{t}$
  - Subcase $X = \alpha$: Prop. 2.6.1 yields $v(X) = v(\alpha_1) \wedge v(\alpha_2)$; hence $v(\alpha_1) = v(\alpha_2) = \mathbf{t}$; extension of Θ is satisfiable
  - Subcase $X = \beta$: Prop. 2.6.1 yields $v(X) = v(\beta_1) \vee v(\beta_2)$; hence $v(\beta_1) = \mathbf{t}$ or $v(\beta_2) = \mathbf{t}$; one of the extensions of Θ is satisfiable
  - Subcases (i) $X = \neg\neg Z$, (ii) $X = \neg\top$, (iii) $X = \neg\perp$: use $v(\neg\neg Z) = v(Z)$, $v(\neg\top) = v(\perp)$ and $v(\neg\perp) = v(\top)$ □

Theorem If there is a closed tableau for S, then S is not satisfiable.

Proof suppose S is satisfiable, a closed tableau exists

- ➡ S is satisfiable, hence the initial one-branch tableau is satisfiable
- ➡ a closed tableau exists, obtained from the inital tableau
- ➡ thus the closed tableau is satisfiable   previous theorem
- ➡ contradiction   definition of satisfiability □

Proof of soundness theorem

- ➡ a tableau proof of X is a closed tableau for $\{\neg X\}$
- ➡ $\{\neg X\}$ is not satisfiable
- ➡ no counter-example exists: X is a tautology □

Definition resolution expansion rules

$$\frac{\neg\neg Z}{Z} \qquad \frac{\neg\top}{\bot} \qquad \frac{\neg\bot}{\top} \qquad \frac{\beta}{\begin{matrix}\beta_1\\ \beta_2\end{matrix}} \qquad \frac{\alpha}{\alpha_1|\alpha_2}$$

Definition resolvent

D_1, D_2 denote disjunctions with X,$\neg X$ occurring as member; the resolvent D of D_1 and D_2 is obtained by

- ➡ deleting all occurrences of X from D_1
- ➡ deleting all occurrences of $\neg X$ from D_2
- ➡ combine the resulting disjunctions

F is called trivial resolvent of D, if obtained by deleting all occurrences of $\bot$ in D

D follows from D_1, D_2 by the resolution rule if D is the (trivial) resolvent of D_1, D_2 (D')

Definition resolution

let $\{A_1, \dots, A_n\}$ be a set of formulas

- ➡ the sequence of disjunctions

$$\begin{matrix}[A_1]\\ [A_2]\\ \vdots\\ [A_n]\end{matrix}$$

  is a resolution expansion for $\{A_1, \dots, A_n\}$
- ➡ suppose $\mathbf{R}$ is a resolution expansion for $\{A_1, \dots, A_n\}$ and $\mathbf{R}^*$ obtained by applying an expansion or resolution rule to $\mathbf{R}$, then $\mathbf{R}^*$ is a resolution expansion for $\{A_1, \dots, A_n\}$

Definition resolution proof

- ➡ a resolution expansion containing the empty clause, is closed
- ➡ a resolution proof of X is a closed resolution expansion for $\{\neg X\}$

Example resolution proof of $P \to (Q \to P)$

$$[\neg(P \to (Q \to P))] \quad [P] \quad [\neg(Q \to P)] \quad [Q] \quad [\neg P] \quad []$$

NB: there is no analog of strictness for the resolution rule, but a sequence of expansion rules is strict, if every disjunction is only expanded once

Theorem soundness of propostional resolution

If X has a resolution proof, then X is a tautology.

Summary

- proof procedures
- propositional semantic tableaux
- tableau implementations
- propositional tableau soundness
- propositional resolution
- propositional resolution soundness