Logic LVA 703600 VU3

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Completeness of a Proof Procedure

recall

- ➡ a branch is closed if X and $\neg X$, or if $\bot$ occur(s) on it
- ➡ a tableau is closed if every branch is closed and a tableau proof of X is a closed tableau for $\{\neg X\}$

the omission is irrelevant for correctness, but critical for completeness

Theorem Completeness for Propositional Tableau

If X is a tautology, X has a tableau proof.

Theorem Completeness for Propositional Resolution

If X is a tautology, X has a resolution proof.

Hintikka's Lemma

Definition propositional Hintikka set

a set $\mathbf{H}$ is a propositional Hintikka set if

- for any propositional letter A, not both $A \in \mathbf{H}$ and $\neg A \in \mathbf{H}$
- $\bot \notin \mathbf{H}$, $\neg\top \notin \mathbf{H}$
- $\neg\neg Z \in \mathbf{H}$ ➟ $Z \in \mathbf{H}$
- $\alpha \in \mathbf{H}$ ➟ $\alpha_1 \in \mathbf{H}$ and $\alpha_2 \in \mathbf{H}$
- $\beta \in \mathbf{H}$ ➟ $\beta_1 \in \mathbf{H}$ or $\beta_2 \in \mathbf{H}$

Example the set $\{P \wedge (\neg Q \to R), P, (\neg Q \to R), \neg\neg Q, Q\}$ is a Hintikka set

Theorem Hintikka's Lemma

Every propositional Hintikka set is satisfiable.

Proof let $\mathbf{H}$ be a Hintikka set; define

$$f(A) = \begin{cases} \mathbf{t} & \text{if } A \in \mathbf{H} \\ \mathbf{f} & \text{if } A \notin \mathbf{H} \end{cases}$$

f uniquely extends to a valuation v (recall Prop. 2.4.2, 2.4.3; Exercise 4.2) such that

- v is well-defined
- $v(X) = \mathbf{t}$ for any $X \in \mathbf{H}$ structural induction

□

Propositional Consistency Property

Definition let $\mathcal{C}$ be a collection of sets; $\mathcal{C}$ is called propositional consistency property if for each $S \in \mathcal{C}$:

- for any propositional letter A, not both $A \in S$ and $\neg A \in S$
- $\bot \notin S$, $\neg\top \notin S$
- $\neg\neg Z \in S$ ➟ $S \cup \{Z\} \in \mathcal{C}$
- $\alpha \in S$ ➟ $S \cup \{\alpha_1, \alpha_2\} \in \mathcal{C}$
- $\beta \in S$ ➟ $S \cup \{\beta_1\} \in \mathcal{C}$ or $S \cup \{\beta_2\} \in \mathcal{C}$

Propositional Model Existence Theorem

Theorem If $\mathcal{C}$ is a propositional consistency property, and $S \in \mathcal{C}$, then S is satisfiable.

proof idea

- show that any $S \in \mathcal{C}$ can be enlarged to $S' \in \mathcal{C}$, such that S' is a Hintikka set
- S' is satisfiable by Hintikka's lemma

proof plan

- show the existence of an extension $\mathcal{C}^*$ of $\mathcal{C}$, such that $\mathcal{C}^*$ is closed under limits (or chain union)
- define a suitable extension **H** of S, such that **H** is a Hintikka set
- apply Hintikka's lemma

Lemma $\mathcal{C}$ is extendable to a (propositional) consistency property $\mathcal{C}^*$ closed under limits: i.e., if $S_1, S_2, \dots \in \mathcal{C}^*$, $S_1 \subseteq S_2 \subseteq \dots$, then $\bigcup_i S_i \in \mathcal{C}^*$

$\mathcal{C}$ a consistency property

$\mathcal{C}$	subset closed	$S \in \mathcal{C}$ implies: for all $S' \subseteq S$, $S' \in \mathcal{C}$
$\mathcal{C}$	of finite character	$S \in \mathcal{C}$ iff for any finite $S' \subseteq S$, $S' \in \mathcal{C}$

Facts:

- ➡ $\mathcal{C}$ is extendable to a consistency property $\mathcal{C}'$ that is subset closed
- ➡ $\mathcal{C}'$ is extendable to a consistency property $\mathcal{C}^*$ of finite character

we show for any finite $S' \subseteq \bigcup_i S_i \in \mathcal{C}^*$: $S' \in \mathcal{C}^*$:

- ➡ let $S' = \{A_1, \dots, A_k\}$; there exists N such that $A_i \in S_N$ for all i
- ➡ hence $S' \subseteq S_N$; as $\mathcal{C}^*$ is subset closed: $S' \in \mathcal{C}^*$ □

let $X_1, X_2, \dots$ be an enumeration of all propositional formulas

$$S_1 := S$$

$$S_{n+1} := \begin{cases} S_n \cup \{X_n\} & \text{if } S_n \cup \{X_n\} \in \mathcal{C}^* \\ S_n & \text{otherwise} \end{cases}$$

$$\mathbf{H} := \bigcup_i S_i$$

- ➡ $S \subseteq \mathbf{H}$ and $\mathbf{H} \in \mathcal{C}^*$
- ➡ $\mathbf{H}$ is maximal in $\mathcal{C}$, i.e. if $\mathbf{H} \subseteq K$ for $K \in \mathcal{C}^*$, then $\mathbf{H} = K$
  - ➡ proof sketch: assume $\mathbf{H} \subsetneq K$, derive a contradiction using that $\mathcal{C}^*$ is subset closed
- ➡ $\mathbf{H}$ is a Hintikka set
  - ➡ proof sketch: suppose $\alpha \in \mathbf{H}$ ➟ $\mathbf{H} \cup \{\alpha_1, \alpha_2\} \in \mathcal{C}^*$ employ maximality ➟ $\alpha_1, \alpha_2 \in \mathbf{H}$
- ➡ $\mathbf{H}$ is satisfiable by Hintikka's lemma, hence S is satisfiable □

Corollaries

Theorem Propositional Compactness

Let S be a set of propositional formulas. If every finite subset of S is satisfiable, so is S.

a formula Z is called interpolant of $X \to Y$ if every propositional letter of Z occurs in X and Y, and $X \to Z$, $Z \to Y$ are tautologies

Example

- $(P \vee (Q \wedge R)) \to (P \vee \neg\neg Q)$ has interpolant $P \vee Q$
- $(P \wedge \neg P) \to Q$ has interpolant $\bot$

Theorem Craig Interpolation

If $X \to Y$ is a tautology, then it has an interpolant.

Tableau Completeness

Definition a finite set S of propositional formulas is tableau consistent if there is no closed tableau for S

Lemma the collection of all tableau consistent sets is a consistency property

Theorem Completeness of Propositional Tableau

If X is a tautology, then X has a tableau proof.

Proof

- suppose X does not have a tableau proof
- hence no closed tableau for $\{\neg X\}$ exists, thus $\{\neg X\}$ is tableau consistent
- hence satisfiable Model Existence Theorem □

Resolution Completeness

let S be a set of disjunctions

- a resolution derivation from S is a sequence of disjunctions, each either a member of S or obtained by an expansion or resolution rule
- let X be a formula; $[X, A_1, \dots, A_n]$ and $[A_1, \dots, A_n]$ are X-enlargements of $[A_1, \dots, A_n]$; the result of replacing each member of S by an X-enlargement, is an X-enlargement of S
- let S_1, S_2 be sets of disjunctions; S_2 an X-enlargement of S_1; if D_1 is resolution derivable from S_1, then there is an X-enlargement D_2 (of D_1) resolution derivable from S_2

Definition a finite set $\{X_1, \dots, X_n\}$ of propositional formulas is resolution consistent if there is no resolution derivation of $[]$ from $\{[X_1], \dots, [X_n]\}$

Lemma the collection of all resolution consistent sets is a consistency property

Proof we verify the conditions of a consistency property

- cases 1), 2) are directly handled by the resolution rule
- case 3, 4) are optional exercises
- case 5: suppose $\beta \in S$, and $S \cup \{\beta_1\}$ and $S \cup \{\beta_2\}$ are both not resolution consistent; suppose $S = \{\beta, X_1, \dots, X_n\}$

  as $S \cup \{\beta_1\}$ is not resolution consistent, there exists a derivation of $[]$ from $\{[X_1], \dots, [X_n], [\beta], [\beta_1]\}$

  in the same way: there exists a derivation of $[]$ from $\{[X_1], \dots, [X_n], [\beta], [\beta_2]\}$

  combining both derivations ➟ there exists a derivation of $[]$ from $\{[X_1], \dots, [X_n], [\beta], [\beta_1, \beta_2]\}$; contradiction □

Theorem Completeness of Propositional Resolution

If X is a tautology, then X has a resolution proof.

NB: completeness with restrictions

- tableau proofs can be restricted to strict tableau, where closure is restricted to atomic formulas
- resolution proofs are restrictable to strict resolution expansion rules, where the resolution rule is only applied to atomic formulas

Definition consequence relation

a formula X is a propositional consequence of a set of formulas S, if X evaluates to $\mathbf{t}$ under each valuation v, such that v satisfies S: we write $S \models_{\mathrm{p}} X$

Fact: $S \models_{\mathrm{p}} X$ iff there is a finite $S_0 \subseteq S$, such that $S_0 \models_{\mathrm{p}} X$

Strong Soundness and Completeness

let S be a set of formulas

- the S-introduction rule for tableau says that any $X \in S$ can be added to any branch
- we write $S \vdash_{\mathrm{pt}} X$ if there is a tableau proof of X admitting S-introduction
- the S-introduction rule for resolution says that any $X \in S$ can be added to a resolution expansion
- we write $S \vdash_{\mathrm{pr}} X$ if there is a closed resolution expansion of $\{\neg X\}$, allowing S-introduction

Theorem For any set S of propositional formulas, and any propositional formula X:

$$S \models_{\mathrm{p}} X \quad \text{iff} \quad S \vdash_{\mathrm{pt}} X \quad \text{iff} \quad S \vdash_{\mathrm{pr}} X \,.$$

Summary

- Hintikka's lemma
- propositional model existence theorem
- completeness of semantic tableau and resolution
- . . . with restrictions
- propositional consequence
- strong soundness and completeness of tableau and resolution