

Logic LVA 703600 VU3

<http://cl-informatik.uibk.ac.at/teaching/ws05/logic/>

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Completeness of a Proof Procedure

recall

- a branch is **closed** if X and $\neg X$, or if \perp occur(s) on it
- a tableau is **closed** if every branch is closed and a **tableau proof** of X is a closed tableau for $\{\neg X\}$

the omission is irrelevant for **correctness**, but critical for **completeness**

Theorem Completeness for Propositional Tableau

If X is a tautology, X has a tableau proof.

Theorem Completeness for Propositional Resolution

If X is a tautology, X has a resolution proof.

Hintikka's Lemma

Definition propositional Hintikka set

a set \mathbf{H} is a **propositional Hintikka set** if

- for any propositional letter A , not both $A \in \mathbf{H}$ and $\neg A \in \mathbf{H}$
- $\perp \notin \mathbf{H}$, $\neg \top \notin \mathbf{H}$
- $\neg\neg Z \in \mathbf{H} \implies Z \in \mathbf{H}$
- $\alpha \in \mathbf{H} \implies \alpha_1 \in \mathbf{H}$ and $\alpha_2 \in \mathbf{H}$
- $\beta \in \mathbf{H} \implies \beta_1 \in \mathbf{H}$ or $\beta_2 \in \mathbf{H}$

Example the set $\{P \wedge (\neg Q \rightarrow R), P, (\neg Q \rightarrow R), \neg\neg Q, Q\}$ is a Hintikka set

Theorem Hintikka's Lemma

Every propositional Hintikka set is satisfiable.

Proof let \mathbf{H} be a Hintikka set; define

$$f(A) = \begin{cases} \mathbf{t} & \text{if } A \in \mathbf{H} \\ \mathbf{f} & \text{if } A \notin \mathbf{H} \end{cases}$$

f uniquely extends to a valuation v (recall Prop. 2.4.2, 2.4.3; Exercise 4.2) such that

- v is well-defined
- $v(X) = \mathbf{t}$ for any $X \in \mathbf{H}$ structural induction

□

Propositional Consistency Property

Definition

let \mathcal{C} be a collection of sets; \mathcal{C} is called **propositional consistency property** if for each $S \in \mathcal{C}$:

- for any propositional letter A , not both $A \in S$ and $\neg A \in S$
- $\perp \notin S$, $\neg\top \notin S$
- $\neg\neg Z \in S \implies S \cup \{Z\} \in \mathcal{C}$
- $\alpha \in S \implies S \cup \{\alpha_1, \alpha_2\} \in \mathcal{C}$
- $\beta \in S \implies S \cup \{\beta_1\} \in \mathcal{C}$ or $S \cup \{\beta_2\} \in \mathcal{C}$

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Propositional Model Existence Theorem

Theorem

If \mathcal{C} is a propositional consistency property, and $S \in \mathcal{C}$, then S is satisfiable.

proof idea

- show that any $S \in \mathcal{C}$ can be enlarged to $S' \in \mathcal{C}$, such that S' is a Hintikka set
- S' is satisfiable by Hintikka's lemma

proof plan

- show the existence of an extension \mathcal{C}^* of \mathcal{C} , such that \mathcal{C}^* is closed under limits (or chain union)
- define a suitable extension \mathbf{H} of S , such that \mathbf{H} is a Hintikka set
- apply Hintikka's lemma

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Lemma

\mathcal{C} is extendable to a (propositional) consistency property \mathcal{C}^* closed under limits: i.e., if $S_1, S_2, \dots \in \mathcal{C}^*$, $S_1 \subseteq S_2 \subseteq \dots$, then $\bigcup_i S_i \in \mathcal{C}^*$

\mathcal{C} a consistency property

- \mathcal{C} subset closed $S \in \mathcal{C}$ implies: for all $S' \subseteq S$, $S' \in \mathcal{C}$
- \mathcal{C} of finite character $S \in \mathcal{C}$ iff for any finite $S' \subseteq S$, $S' \in \mathcal{C}$

Facts:

- \mathcal{C} is extendable to a consistency property \mathcal{C}' that is subset closed
- \mathcal{C}' is extendable to a consistency property \mathcal{C}^* of finite character

we show for any finite $S' \subseteq \bigcup_i S_i \in \mathcal{C}^*$: $S' \in \mathcal{C}^*$:

- let $S' = \{A_1, \dots, A_k\}$; there exists N such that $A_i \in S_N$ for all i
- hence $S' \subseteq S_N$; as \mathcal{C}^* is subset closed: $S' \in \mathcal{C}^*$

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let X_1, X_2, \dots be an enumeration of all propositional formulas

$$S_1 := S$$

$$S_{n+1} := \begin{cases} S_n \cup \{X_n\} & \text{if } S_n \cup \{X_n\} \in \mathcal{C}^* \\ S_n & \text{otherwise} \end{cases}$$

$$\mathbf{H} := \bigcup_i S_i$$

- $S \subseteq \mathbf{H}$ and $\mathbf{H} \in \mathcal{C}^*$
- \mathbf{H} is maximal in \mathcal{C} , i.e. if $\mathbf{H} \subseteq K$ for $K \in \mathcal{C}^*$, then $\mathbf{H} = K$
 - proof sketch: assume $\mathbf{H} \subsetneq K$, derive a contradiction using that \mathcal{C}^* is subset closed
- \mathbf{H} is a Hintikka set
 - proof sketch: suppose $\alpha \in \mathbf{H} \implies \mathbf{H} \cup \{\alpha_1, \alpha_2\} \in \mathcal{C}^*$ employ maximality $\implies \alpha_1, \alpha_2 \in \mathbf{H}$
- \mathbf{H} is satisfiable by Hintikka's lemma, hence S is satisfiable \square

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Corollaries

Theorem Propositional Compactness

Let S be a set of propositional formulas. If every finite subset of S is satisfiable, so is S .

a formula Z is called **interpolant** of $X \rightarrow Y$ if every propositional letter of Z occurs in X and Y , and $X \rightarrow Z$, $Z \rightarrow Y$ are tautologies

Example

- $(P \vee (Q \wedge R)) \rightarrow (P \vee \neg\neg Q)$ has interpolant $P \vee Q$
- $(P \wedge \neg P) \rightarrow Q$ has interpolant \perp

Theorem Craig Interpolation

If $X \rightarrow Y$ is a tautology, then it has an interpolant.

Tableau Completeness

Definition a finite set S of propositional formulas is **tableau consistent** if there is no closed tableau for S

Lemma the collection of all tableau consistent sets is a consistency property

Theorem Completeness of Propositional Tableau

If X is a tautology, then X has a tableau proof.

Proof

- suppose X does not have a tableau proof
- hence no closed tableau for $\{\neg X\}$ exists, thus $\{\neg X\}$ is tableau consistent
- hence satisfiable Model Existence Theorem \square

Resolution Completeness

let S be a set of disjunctions

- a **resolution derivation** from S is a sequence of disjunctions, each either a member of S or obtained by an expansion or resolution rule
- let X be a formula; $[X, A_1, \dots, A_n]$ and $[A_1, \dots, A_n]$ are **X -enlargements** of $[A_1, \dots, A_n]$; the result of replacing each member of S by an X -enlargement, is an X -enlargement of S
- let S_1, S_2 be sets of disjunctions; S_2 an X -enlargement of S_1 ; if D_1 is resolution derivable from S_1 , then there is an X -enlargement D_2 (of D_1) resolution derivable from S_2

Definition a finite set $\{X_1, \dots, X_n\}$ of propositional formulas is **resolution consistent** if there is no resolution derivation of \square from $\{[X_1], \dots, [X_n]\}$

Lemma the collection of all resolution consistent sets is a consistency property

Proof

- we verify the conditions of a consistency property
- cases 1), 2) are directly handled by the resolution rule
- case 3, 4) are optional exercises
- case 5: suppose $\beta \in S$, and $S \cup \{\beta_1\}$ and $S \cup \{\beta_2\}$ are both not resolution consistent; suppose $S = \{\beta, X_1, \dots, X_n\}$
 as $S \cup \{\beta_1\}$ is not resolution consistent, there exists a derivation of \square from $\{[X_1], \dots, [X_n], [\beta], [\beta_1]\}$
 in the same way: there exists a derivation of \square from $\{[X_1], \dots, [X_n], [\beta], [\beta_2]\}$
 combining both derivations \Rightarrow there exists a derivation of \square from $\{[X_1], \dots, [X_n], [\beta], [\beta_1], [\beta_2]\}$; contradiction \square

Theorem**Completeness of Propositional Resolution**

If X is a tautology, then X has a resolution proof.

NB: completeness with restrictions

- ➔ tableau proofs can be restricted to strict tableau, where closure is restricted to atomic formulas
- ➔ resolution proofs are restrictable to strict resolution expansion rules, where the resolution rule is only applied to atomic formulas

Definition**consequence relation**

a formula X is a **propositional consequence** of a set of formulas S , if X evaluates to **t** under each valuation v , such that v satisfies S : we write $S \models_p X$

Fact: $S \models_p X$ iff there is a finite $S_0 \subseteq S$, such that $S_0 \models_p X$

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**Strong Soundness and Completeness**

let S be a set of formulas

- ➔ the **S -introduction rule for tableau** says that any $X \in S$ can be added to any branch
- ➔ we write $S \vdash_{pt} X$ if there is a tableau proof of X admitting S -introduction
- ➔ the **S -introduction rule for resolution** says that any $X \in S$ can be added to a resolution expansion
- ➔ we write $S \vdash_{pr} X$ if there is a closed resolution expansion of $\{\neg X\}$, allowing S -introduction

Theorem

For any set S of propositional formulas, and any propositional formula X :

$$S \models_p X \quad \text{iff} \quad S \vdash_{pt} X \quad \text{iff} \quad S \vdash_{pr} X .$$

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**Summary**

- ➔ Hintikka's lemma
- ➔ propositional model existence theorem
- ➔ completeness of semantic tableau and resolution
- ➔ ... with restrictions
- ➔ propositional consequence
- ➔ strong soundness and completeness of tableau and resolution

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