Logic LVA 703600 VU3

http://cl-informatik.uibk.ac.at/teaching/ws05/logic/

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Definition first-order language

fixed part	connectives	as for propositional logic
	quantifiers	$\forall$ for all
		$\exists$ there exists
	auxiliary symbols	')','(', and ','
	variables	$v_1, v_2, \ldots$
		informally: $x, y, z, \ldots$
variable part	relation symbols	$\mathbf{R}$
	function symbols	$\mathbf{F}$
	constant symbols	$\mathbf{C}$

notation $\mathsf{L}(\mathbf{R}, \mathbf{F}, \mathbf{C})$ describes the first-order language determined by $\mathbf{R}, \mathbf{F}, \mathbf{C}$

Definition inductive definition of terms & formulas

terms, **T**	any variable is in **T**	
	any constant is in **T**	
	$t_1, \dots, t_n \in \mathbf{T}$ ➟	
	➟ $f(t_1, \dots, t_n) \in \mathbf{T}$	$f \in \mathbf{F}$, n-ary
atomic formulas	$R(t_1, \dots, t_n)$	$R \in \mathbf{R}$, n-ary
	$\top$, $\bot$	$t_1, \dots, t_n \in \mathbf{T}$
formulas, **Frm**	any atomic formula is in **Frm**	
	$A \in \mathbf{Frm}$ ➟ $\neg A \in \mathbf{Frm}$	
	$A, B \in \mathbf{Frm}$ ➟ $(A \circ B) \in \mathbf{Frm}$	$\circ$ binary
	$A \in \mathbf{Frm}$ ➟ $(\forall x)A \in \mathbf{Frm}$	x a variable
	$A \in \mathbf{Frm}$ ➟ $(\exists x)A \in \mathbf{Frm}$	

Example formulas

$$(\forall x)(\forall y)(< (x,y) \rightarrow (\exists z)(< (x,z) \wedge \ < (z,y)))$$
$$(\forall x)(\forall y)[x < y \rightarrow (\exists z)(x < z \wedge z < y)]$$
$$\forall x, y \ x < y \rightarrow \exists z \ (x < z \wedge z < y)$$

NB: we employ structural induction on terms (and formulas) and structural recursion on terms (and formulas) as proof-principles

Example free-variable occurrences, fvar(X)

- ➡ if A atomic, then fvar(A) is the set of variables occurring in A
- ➡ $\mathrm{fvar}(\neg A) := \mathrm{fvar}(A)$
- ➡ $\mathrm{fvar}((A \circ B)) := \mathrm{fvar}(A) \cup \mathrm{fvar}(B)$
- ➡ $\mathrm{fvar}((\forall x)A) = \mathrm{fvar}((\exists x)A) := \mathrm{fvar}(A) - \{x\}$

Definition

substitutions

- a substitution σ is a mapping $\sigma\colon \mathbf{V} \to \mathbf{T}$ from the set of variables to the set of terms $\mathbf{T}$
- extend σ to terms:

$$
\begin{aligned}
c\sigma &:= c && c \in \mathbf{C} \\
f(t_1,\dots,t_n)\sigma &:= f(t_1\sigma,\dots,t_n\sigma) && f \in \mathbf{F}
\end{aligned}
$$

example: set σ: $x \mapsto f(x,y), y \mapsto h(a), z \mapsto g(c,h(x))$

$$j(k(x),y)\sigma = j(k(x)\sigma, y\sigma) = j(k(x\sigma), y\sigma) = j(k(f(x,y)), h(a))$$

- σ, τ substitutions; the composition $\sigma\tau$ of σ and τ is defined as

$$x(\sigma\tau) := (x\sigma)\tau$$

- the domain of σ is $\{x \mid x\sigma \neq x\}$

Lemma

- for any term t: $t(\sigma\tau) = (t\sigma)\tau$ by structural induction
- composition is associative (i.e. $(\sigma_1\sigma_2)\sigma_3 = \sigma_1(\sigma_2\sigma_3)$)

Definition

- if the domain of σ is $\{x_1,\dots,x_n\}$ and $x_1\sigma = t_1$,...,$x_n\sigma = t_n$, then we write $\{x_1 \mapsto t_1,\dots,x_n \mapsto t_n\}$ to denote σ

Theorem

set $\sigma_1 = \{x_1 \mapsto t_1,\dots,x_n \mapsto t_n\}$, $\sigma_2 = \{y_1 \mapsto s_1,\dots,y_k \mapsto s_k\}$; then the composition $\sigma_1\sigma_2$ can be written as

$$\{x_1 \mapsto t_1\sigma_2,\dots,x_n \mapsto t_n\sigma_2, z_1 \mapsto z_1\sigma_2,\dots,z_m \mapsto z_m\sigma_2\}$$

$$\{z_1,\dots,z_m\} = \{y_1,\dots,y_k\} - \{x_1,\dots,x_n\}$$

Definition substitutions on formulas

let σ be a substitution

➡ define σ_x:

$$y\sigma_x := \begin{cases} y\sigma & \text{if } y \neq x \\ x & \text{otherwise} \end{cases}$$

➡ $P(t_1, \ldots, t_n)\sigma := P(t_1\sigma, \ldots, t_n\sigma)$ $\quad P \in \mathbf{R}$, P n-ary

$\top\sigma := \top \qquad \bot\sigma := \bot$

$(\neg A)\sigma := \neg(A\sigma)$

$(A \circ B)\sigma := (A\sigma \circ B\sigma)$ $\quad \circ$ binary

$((\forall x)A)\sigma := (\forall x)(A\sigma_x)$

$((\exists x)A)\sigma := (\exists x)(A\sigma_x)$

Example $\sigma = \{x \mapsto a, y \mapsto b\}$

$$\begin{aligned}(\forall x R(x,y) \to \exists y R(x,y))\sigma &= (\forall x R(x,y))\sigma \to (\exists y R(x,y))\sigma \\ &= \forall x(R(x,y)\sigma_x) \to \exists y(R(x,y)\sigma_y) \\ &= \forall x R(x,b) \to \exists y R(a,y)\end{aligned}$$

Definition substitution σ is free for a formula:

➡ if A atomic, σ is free for A

➡ σ is free for A ➟ σ is free for $\neg A$

➡ σ is free for A and B ➟ σ is free for $(A \circ B)$

➡ σ_x is free for A and if $y \in \mathrm{fvar}(A)$, $y \neq x$, then $y\sigma$ does not contain x ➟ σ is free for $(\exists x)A$ and free for $(\forall x)A$

Theorem Suppose σ is free for A and τ is free for $A\sigma$, then $(A\sigma)\tau = A(\sigma\tau)$

Proof by structural induction on A

- Base: let $A = P(t_1, \ldots, t_n)$, hence

$$(P(t_1, \ldots, t_n)\sigma)\tau = P((t_1\sigma)\tau, \ldots, (t_n\sigma)\tau) =$$
$$= P(t_1(\sigma\tau), \ldots, t_n(\sigma\tau)) = P(\sigma\tau)$$

- Step: we only consider $A = (\forall x)A_1$

assumptions σ_x is free for A_1

τ_x is free for $A_1\sigma_x$ as $((\forall x)A_1)\sigma = (\forall x)A_1\sigma_x$

$(A_1\sigma_x)\tau_x = A_1(\sigma_x\tau_x)$ by IH

$A_1(\sigma_x\tau_x) = A_1(\sigma\tau)_x$ easy

$$(((\forall x)A_1)\sigma)\tau = ((\forall x)(A_1\sigma_x))\tau = (\forall x)((A_1\sigma_x)\tau_x) =$$
$$= (\forall x)(A_1(\sigma_x\tau_x)) = (\forall x)(A_1(\sigma\tau)_x) = ((\forall x)A_1)(\sigma\tau) \quad \square$$

Definition domain, models & assignments

model: a model of $L(\mathbf{R}, \mathbf{F}, \mathbf{C})$ is a pair $\mathbf{M} = (\mathbf{D}, \mathbf{I})$, s.t.

$\mathbf{D} \neq \emptyset$ a set, called domain of $\mathbf{M}$

$\mathbf{I}$ a mapping, called interpretation associating

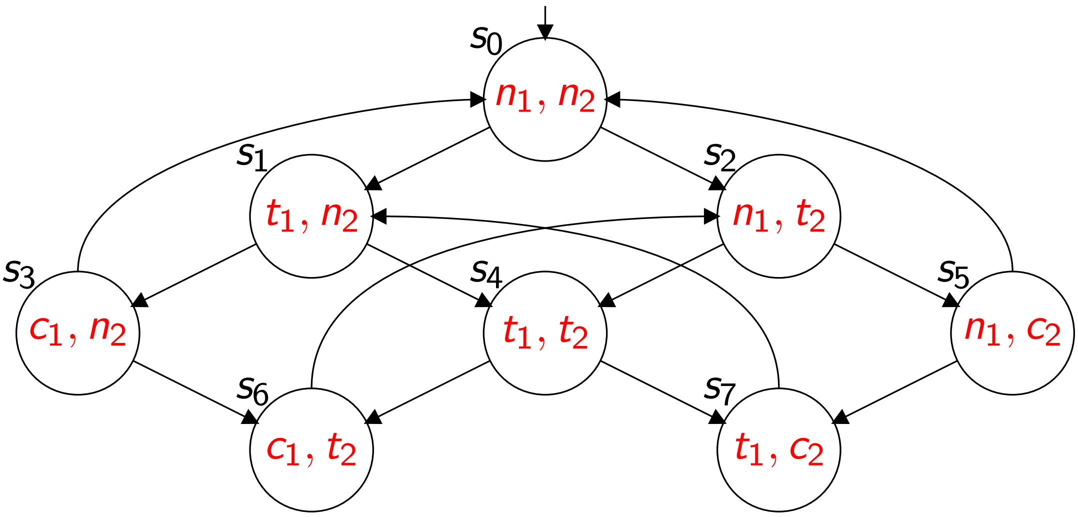
- to every $c \in \mathbf{C}$, some $c^{\mathbf{I}} \in \mathbf{D}$
- to every $f \in \mathbf{F}$, some function $f^{\mathbf{I}} \colon \mathbf{D}^n \to \mathbf{D}$
- to every $P \in \mathbf{R}$, some relation $P^{\mathbf{I}} \subseteq \mathbf{D}^n$

assignment: an assignment in $\mathbf{M}$ is a mapping $\mathbf{A} \colon \mathbf{V} \to \mathbf{D}$; we write $v^{\mathbf{A}}$ instead of $A(v)$

value: each term $t \in L$ is associated a value $t^{\mathbf{I},\mathbf{A}}$

- for $c \in \mathbf{C}$: $c^{\mathbf{I},\mathbf{A}} := c^{\mathbf{I}}$
- for $v \in \mathbf{V}$: $v^{\mathbf{I},\mathbf{A}} := v^{\mathbf{A}}$
- for $f \in \mathbf{F}$: $[f(t_1, \ldots, t_n)]^{\mathbf{I},\mathbf{A}} := f^{\mathbf{I}}(t_1^{\mathbf{I},\mathbf{A}}, \ldots, t_n^{\mathbf{I},\mathbf{A}})$

Example: Modelling 'mutual exclusion'



define a first-order language $\mathsf{L}(\mathbf{R}, \mathbf{F}, \mathbf{C})$ for 'mutual exclusion'

relation symbols	R binary
	C_i, N_i, T_i unary for $i \in [1, 2]$
function symbols	none
constant symbols	$k_0, k_1, \ldots, k_7$

represent the protocol by a first-order model $\mathbf{M} = (\mathbf{D}, \mathbf{I})$

domain: $\mathbf{D} = \{s_0, s_1, \ldots, s_7\}$

interpretation $\mathbf{I}$:

- $\mathbf{I}$ interprets the symbol R by the relation $R^{\mathbf{I}}$ where

$$\{(s_0, s_1), (s_0, s_2), (s_1, s_3), (s_1, s_4), (s_2, s_4), (s_2, s_5), (s_3, s_6),$$
$$(s_4, s_6), (s_4, s_7), (s_5, s_7), (s_6, s_2), (s_7, s_1), (s_3, s_0), (s_5, s_0)\} = R^{\mathbf{I}}$$

- $\mathbf{I}$ interprets C_i by $C_i^{\mathbf{I}}$ where $s_3, s_6 \in C_1^{\mathbf{I}}$ and $s_5, s_7 \in C_2^{\mathbf{I}}$
- $\mathbf{I}$ interprets N_i by $N_i^{\mathbf{I}}$ where $s_0, s_2, s_5 \in N_1^{\mathbf{I}}$ and $s_0, s_1, s_3 \in N_2^{\mathbf{I}}$
- $\mathbf{I}$ interprets T_i by $T_i^{\mathbf{I}}$ where $s_1, s_4, s_7 \in T_1^{\mathbf{I}}$ and $s_2, s_4, s_6 \in T_2^{\mathbf{I}}$
- finally $\mathbf{I}$ associates a state s_i with each c_i: $c_i^{\mathbf{I}} = s_i$

assignment $\mathbf{A}$: arbitrary

Definition truth value for formulas

- an assignment $\mathbf{B}$ in a model $\mathbf{M}$ is an *x-variant* of an assignment $\mathbf{A}$ if the values differ only for x
- let $\mathbf{M} = (\mathbf{D}, \mathbf{I})$ be a model, $\mathbf{A}$ an assignment in $\mathbf{M}$; define the truth-value $[X]^{\mathbf{I},\mathbf{A}}$ of a formula X
  - $[P(t_1, \ldots, t_n)]^{\mathbf{I},\mathbf{A}} = \mathbf{t}$ iff $(t_1^{\mathbf{I},\mathbf{A}}, \ldots, t_n^{\mathbf{I},\mathbf{A}}) \in P^{\mathbf{I}}$
    $\top^{\mathbf{I},\mathbf{A}} := \mathbf{t}$
    $\bot^{\mathbf{I},\mathbf{A}} := \mathbf{f}$
  - $[\neg A]^{\mathbf{I},\mathbf{A}} := \neg(A^{\mathbf{I},\mathbf{A}})$
  - $[(A \circ B)]^{\mathbf{I},\mathbf{A}} := (A^{\mathbf{I},\mathbf{A}} \circ B^{\mathbf{I},\mathbf{A}})$
  - $[(\forall x)A]^{\mathbf{I},\mathbf{A}} = \mathbf{t}$ iff $A^{\mathbf{I},\mathbf{B}} = \mathbf{t}$ for every $\mathbf{B}$, $\mathbf{B}$ x-variant of $\mathbf{A}$
  - $[(\exists x)A]^{\mathbf{I},\mathbf{A}} = \mathbf{t}$ iff $A^{\mathbf{I},\mathbf{B}} = \mathbf{t}$ for some $\mathbf{B}$, $\mathbf{B}$ x-variant of $\mathbf{A}$

Definition validity & satisfiability

- X is *true in* $\mathbf{M}$, if $X^{\mathbf{I},\mathbf{A}} = \mathbf{t}$ for all assignments $\mathbf{A}$
- X is *valid*, if X is true in all models for the language
- as set S of formulas is *satisfiable in* $\mathbf{M}$, if there is some $\mathbf{A}$ such that $X^{\mathbf{I},\mathbf{A}} = \mathbf{t}$ for all $X \in S$
- S is *satisfiable*, if satisfiable in some $\mathbf{M}$

Example let $\mathbf{M}$ be the model of the 'mutual exclusion' protocol; we show that $\forall x \neg(C_1(x) \wedge C_2(x))$ in true in $\mathbf{M}$

$[\forall x \neg(C_1(x) \wedge C_2(x))]^{\mathbf{I},\mathbf{A}} = \mathbf{t}$

iff $[\neg(C_1(x) \wedge C_2(x))]^{\mathbf{I},\mathbf{B}} = \mathbf{t}$ for any x-variant $\mathbf{B}$ of $\mathbf{A}$

iff $\neg([C_1(x)]^{\mathbf{I},\mathbf{B}} \wedge [C_2(x)]^{\mathbf{I},\mathbf{B}}) = \mathbf{t}$ for any x-variant $\mathbf{B}$ of $\mathbf{A}$

iff $\neg(x^{\mathbf{B}} \in C_1^{\mathbf{I}} \wedge x^{\mathbf{B}} \in C_2^{\mathbf{I}}) = \mathbf{t}$ for any x-variant $\mathbf{B}$ of $\mathbf{A}$

iff $\neg(s \in C_1^{\mathbf{I}} \wedge s \in C_2^{\mathbf{I}}) = \mathbf{t}$ for any $s \in \mathbf{D}$

Summary

- ➡ syntax of first-order logic
- ➡ substitutions
- ➡ semantics of first-order logic