Logic LVA 703600 VU3

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Georg Moser (VU)[1] Christian Vogt (VU)[2]

[1]georg.moser@uibk.ac.at
office hours: Thursday 1pm–3pm

[2]christian.vogt@uibk.ac.at
office hours: Tuesday 9am–11am

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Substitutions & Assignments

Theorem Suppose t is closed, X is a formula of L, $\mathbf{M} = (\mathbf{D}, \mathbf{I})$ is a model for L. Let x be a variable and let $\mathbf{A}$ be an assignment s.t. $x^{\mathbf{A}} = t^{\mathbf{I}}$. Then $[X\{x \mapsto t\}]^{\mathbf{I},\mathbf{A}} = X^{\mathbf{I},\mathbf{A}}$. More generally, if $\mathbf{B}$ is an x-variant of $\mathbf{A}$, then $[X\{x \mapsto t\}]^{\mathbf{I},\mathbf{B}} = X^{\mathbf{I},\mathbf{A}}$.

Proof by structural induction on X □

Theorem Let $\mathbf{M}$, X, and $\mathbf{A}$ be as above. Let σ be a substitution free for X. Define $\mathbf{B}$ by setting for each variable $v^{\mathbf{B}} = (v\sigma)^{\mathbf{I},\mathbf{A}}$. Then $X^{\mathbf{I},\mathbf{B}} = (X\sigma)^{\mathbf{I},\mathbf{A}}$.

Proof by structural induction on X using

- for any term t: $t^{\mathbf{I},\mathbf{B}} = (t\sigma)^{\mathbf{I},\mathbf{A}}$
- σ is free for X □

Herbrand Models

Definition Herbrand model

a model $\mathbf{M} = (\mathbf{D}, \mathbf{I})$ for the language L is a Herbrand model, if

- $\mathbf{D}$ is the set of closed terms (of L)
- for each closed term t: $t^{\mathbf{I}} = t$

Example consider L with $\mathbf{R} = \{P\}$, $\mathbf{F} = \{S\}$, $\mathbf{C} = \{0\}$, then the infinite set

$$\{P(0), P(S^2(0)), \ldots, P(S^{2n}(0)), P(S^{2n+2}(0)), \ldots\}$$

uniquely describes a Herbrand model $\mathbf{M}$:

- $\mathbf{D} = \{0, S(0), S(S(0)), \ldots\}$
- for $d \in \mathbf{D}$, $S^{\mathbf{I}}(d) := S(d)$ and for n even, we set $S^n(0) \in P^{\mathbf{I}}$
- $\mathbf{I}$ associates $0 \in \mathbf{D}$ with 0, S^I with S, $P^{\mathbf{I}}$ with P

Theorem if $\mathbf{M} = (\mathbf{D}, \mathbf{I})$ is a Herbrand model for L, then for any term t: $t^{\mathbf{I},\mathbf{A}} = (t\mathbf{A})^{\mathbf{I}}$

Proof by induction on t □

Theorem if $\mathbf{M} = (\mathbf{D}, \mathbf{I})$ is a Herbrand model for L, then for any formula X: $X^{\mathbf{I},\mathbf{A}} = (X\mathbf{A})^{\mathbf{I}}$

Theorem if A a formula, $\mathbf{M} = (\mathbf{D}, \mathbf{I})$ a Herbrand model, then

- $(\forall x)A$ is true in $\mathbf{M}$ iff $A\{x \mapsto d\}$ is true in $\mathbf{M}$ for every $d \in \mathbf{D}$
- $(\exists x)A$ is true in $\mathbf{M}$ iff $A\{x \mapsto d\}$ is true in $\mathbf{M}$ for some $d \in \mathbf{D}$

Uniform Notation

quantified formulas are grouped into two categories:

universal

γ	$\gamma(t)$
$(\forall x)A$	$A\{x \mapsto t\}$
$\neg(\exists x)A$	$\neg A\{x \mapsto t\}$

existential

δ	$\delta(t)$
$(\exists x)A$	$A\{x \mapsto t\}$
$\neg(\forall x)A$	$\neg A\{x \mapsto t\}$

we obtain

- ➡ $\gamma \equiv (\forall y)\gamma(y)$ and $\delta \equiv (\exists y)\delta(y)$, provided y is new

Theorem suppose $\mathbf{M} = (\mathbf{D}, \mathbf{I})$ is a Herbrand model for L

- ➡ γ is true in $\mathbf{M}$ iff $\gamma(d)$ is true in $\mathbf{M}$ for every $d \in \mathbf{D}$
- ➡ δ is true in $\mathbf{M}$ iff $\delta(d)$ is true in $\mathbf{M}$ for some $d \in \mathbf{D}$

Hintikka's Lemma

Definition first-order Hintikka set

let L be a language; a set $\mathbf{H}$ of sentences is a first-order Hintikka set (wrt L) if

- ➡ for any atomic formula A, not both $A \in \mathbf{H}$ and $\neg A \in \mathbf{H}$
- ➡ $\bot \notin \mathbf{H}$, $\neg\top \notin \mathbf{H}$
- ➡ $\neg\neg Z \in \mathbf{H}$ ➟ $Z \in \mathbf{H}$
- ➡ $\alpha \in \mathbf{H}$ ➟ $\alpha_1 \in \mathbf{H}$ and $\alpha_2 \in \mathbf{H}$
- ➡ $\beta \in \mathbf{H}$ ➟ $\beta_1 \in \mathbf{H}$ or $\beta_2 \in \mathbf{H}$
- ➡ $\gamma \in \mathbf{H}$ ➟ $\gamma(t) \in \mathbf{H}$ for every closed term t of L
- ➡ $\delta \in \mathbf{H}$ ➟ $\delta(t) \in \mathbf{H}$ for some closed term t of L

Example if L is defined as $\mathbf{R} = \{P, Q\}$, $\mathbf{F} = \{S\}$, $\mathbf{C} = \{0\}$, then

$$\{\}$$

$$\{P(0), P(S^2(0)), \ldots, P(S^n(0)), P(S^{n+2}(0)), \ldots\}$$

$$\{((\exists x)Q(x) \vee \neg(\exists y)P(y)), (\exists x)Q(x), Q(S(S(0))), \neg(\exists y)P(y), \neg P(0), \neg P(S(0)), \ldots\}$$

are Hintikka sets wrt L, but

$$\{((\forall y)P(y) \wedge (\exists x)Q(x)), (\forall y)P(y), (\exists x)Q(x), Q(0)\}$$

is not

Lemma Hintikka's lemma

Suppose L is a language with a nonempty set of closed terms. If **H** is a first-order Hintikka set wrt L, then **H** is satisfiable in a Herbrand model

Proof first, we construct a model $\mathbf{M} = (\mathbf{D}, \mathbf{I})$

- **D** is the set of closed terms
- for each $c \in \mathcal{C}$, we set $c^{\mathbf{I}} = c$
- for each $f \in \mathbf{F}$ and $t_1, \ldots, t_n \in \mathbf{D}$, we set $f^{\mathbf{I}}(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$
- for each $P \in \mathbf{R}$, we set $(t_1, \ldots, t_n) \in P^{\mathbf{I}}$ if $P(t_1, \ldots, t_n) \in \mathbf{H}$

we obtain $t^{\mathbf{I}} = t$ for each closed t; hence **M** is a Herbrand model; now we prove by induction on A: if $A \in \mathbf{H}$, then A is true in **M**

- Base suppose $P(t_1, \ldots, t_n)$ is an atomic sentence in **H**; let **A** be an arbitrary assignment

$(t_1, \ldots, t_n) \in P^{\mathbf{I}}$	by definition of **M**
$(t_1^{\mathbf{I},\mathbf{A}}, \ldots, t_n^{\mathbf{I},\mathbf{A}}) \in P^{\mathbf{I}}$	for each i: $t_i^{\mathbf{I},\mathbf{A}} = t_i^{\mathbf{I}} = t_i$
$[P(t_1, \ldots, t_n)]^{\mathbf{I},\mathbf{A}} = \mathbf{t}$	

- Step follows directly from IH and the previous theorem □

Definition let $L(\mathbf{R}, \mathbf{F}, \mathbf{C})$ be a language, **par** be a countable set of constants, distinct from **C**;

- ➡ the members of **par** are called parameters
- ➡ we write L^{par} for $L(\mathbf{R}, \mathbf{F}, \mathbf{C} \cup \mathbf{par})$

Definition let $\mathcal{C}$ be a collection of sets of sentences of L^{par}, it is called first-order consistency property (FCP) if for each $S \in \mathcal{C}$:

- ➡ for any atomic sentence A, not both $A \in S$ and $\neg A \in S$
- ➡ $\bot \notin S$, $\neg\top \notin S$
- ➡ $\neg\neg Z \in S$ ➡ $S \cup \{Z\} \in \mathcal{C}$
- ➡ $\alpha \in S$ ➡ $S \cup \{\alpha_1, \alpha_2\} \in \mathcal{C}$
- ➡ $\beta \in S$ ➡ $S \cup \{\beta_1\} \in \mathcal{C}$ or $S \cup \{\beta_2\} \in \mathcal{C}$
- ➡ $\gamma \in S$ ➡ $S \cup \{\gamma(t)\} \in \mathcal{C}$ for every closed term t of L^{par}
- ➡ $\delta \in S$ ➡ $S \cup \{\delta(p)\} \in \mathcal{C}$ for some parameter p of L^{par}

First-Order Model Existence

Theorem first-order model existence

If $\mathcal{C}$ is a FCP wrt L, S is a set of sentences(!) of L, and $S \in \mathcal{C}$, then S is satisfiable in a Herbrand model (a Herbrand model wrt L^{par})

Proof let $\mathcal{C}$ be a FCP

$\mathcal{C}$	subset closed	$S \in \mathcal{C}$ implies: for all $S_0 \subseteq S$, $S_0 \in \mathcal{C}$
$\mathcal{C}$	of finite character	$S \in \mathcal{C}$ iff for any finite $S' \subseteq S$, $S' \in \mathcal{C}$

proof plan

- ➡ show the existence of an extension $\mathcal{C}^*$ of $\mathcal{C}$, such that $\mathcal{C}^*$ is a FCP of finite character
- ➡ define a suitable extension **H** of S, such that **H** is a (first-order) Hintikka set
- ➡ apply Hintikka's lemma

Lemma let $\mathcal{C}$ be a FCP, then $\mathcal{C}$ is extendable to a FCP $\mathcal{C}^+$, such that $\mathcal{C}^+$ is subset closed

Proof let $\mathcal{C}^+$ consist of all subsets of members of $\mathcal{C}$, then the lemma follows easily □

moreover let $\mathcal{C}^*$ be an extension of $\mathcal{C}$, we would have to show that $\mathcal{C}^*$ admits: $\delta \in S \Rightarrow S \cup \{\delta(p)\} \in \mathcal{C}^*$ for some parameter p of L^{par}

- problem: does not work!
- solution: change the definition of FCP

Definition alternate first-order consistency property

let

- $\delta \in S \Rightarrow S \cup \{\delta(p)\} \in \mathcal{C}$ for any parameter p that is new to S

replace the last rule in the definition of a FCP

Substitutions, Assignments Herbrand models Uniform notation Hintikka's lemma Model Existence

Definition parameter substitution

a parameter substitution π is a mapping $\pi\colon \mathbf{par} \to \mathbf{par}$, π is extended to a substitution on formulas (of L^{par}) and sets of formulas

Lemma let $\mathcal{C}$ be a FCP, closed under subsets; define $\mathcal{C}^+$ as follows: if $S\pi \in \mathcal{C}$ for some parameter substitution, then $S \in \mathcal{C}^+$; we obtain

- $\mathcal{C}^+$ extends $\mathcal{C}$
- $\mathcal{C}^+$ is closed under subsets
- $\mathcal{C}^+$ is an alternate FCP

Proof we clarify the first point: we have to show that $S \in \mathcal{C}$ implies $S \in \mathcal{C}^+$

- $S \in \mathcal{C}^+$ if $S\pi \in \mathcal{C}$ for some parameter substitution
- let π be the identity substitution □

Lemma let $\mathcal{C}^+$ be an alternate FCP that is subset closed, then

- ➡ $\mathcal{C}^+$ is extendable to a $\mathcal{C}^*$ of finite character
- ➡ $\mathcal{C}^*$ is subset closed

Proof let $\mathcal{C}^*$ consist of those sets, all of whose finite subsets are in $\mathcal{C}$; the lemma follows easily □

let $X_1, X_2, \ldots$ be an enumeration of all first-order sentences

$$S_1 := S$$

$$S_{n+1} := \begin{cases} S_n & \text{if } S_n \cup \{X_n\} \notin \mathcal{C}^* \\ S_n \cup \{X_n\} & \text{if } S_n \cup \{X_n\} \in \mathcal{C}^* \text{ and } X_n \text{ is not } \delta \\ S_n \cup \{X_n\} \cup \{\delta(p)\} & \text{if } S_n \cup \{X_n\} \in \mathcal{C}^*,\ X_n \text{ is } \delta, \\ & p \text{ a new parameter} \end{cases}$$

NB: each S_i leaves infinitely many parameters unused

finally set

$$\mathbf{H} := \bigcup_i S_i$$

by construction $S \in \mathbf{H}$, furthermore

- ➡ $\mathbf{H} \in \mathcal{C}^*$: follows from the fact that $\mathcal{C}^*$ is of finite character
- ➡ **H** is maximal: follows by definition
- ➡ **H** is a first-order Hintikka set: we consider two cases:
  - ➡ suppose $\gamma \in \mathbf{H}$
    $\mathbf{H} \cup \{\gamma(t)\} \in \mathcal{C}^*$ for every closed term t
    $\mathbf{H} \cup \{\gamma(t)\} = \mathbf{H}$
  - ➡ suppose $\delta \in \mathbf{H}$
    $\delta \in \mathbf{H}$ implies $\delta(p) \in \mathbf{H}$ for some parameter p (by definition)
    hence $\delta(t) \in \mathbf{H}$ for some term t in $\mathrm{L}^{\mathrm{par}}$
- ➡ **H** is satisfiable by Hintikka's lemma
- ➡ note that **H** is satisfiable by a Herbrand model wrt $\mathrm{L}^{\mathrm{par}}$ □

Summary

- substitutions and assignments
- Herbrand models
- uniform notation for first-order formulas
- Hintikka's lemma
- first-order model existence