

# Logic LVA 703600 VU3

<http://cl-informatik.uibk.ac.at/teaching/ws05/logic/>

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## Herbrand Models

### Definition

#### Herbrand model

a model  $\mathbf{M} = (\mathbf{D}, \mathbf{I})$  for the language  $L$  is a Herbrand model, if

- $\mathbf{D}$  is the set of closed terms (of  $L$ )
- for each closed term  $t$ :  $t^{\mathbf{I}} = t$

### Example

consider  $L$  with  $\mathbf{R} = \{P\}$ ,  $\mathbf{F} = \{S\}$ ,  $\mathbf{C} = \{0\}$ , then the infinite set

$$\{P(0), P(S^2(0)), \dots, P(S^{2n}(0)), P(S^{2n+2}(0)), \dots\}$$

uniquely describes a Herbrand model  $\mathbf{M}$ :

- $\mathbf{D} = \{0, S(0), S(S(0)), \dots\}$
- for  $d \in \mathbf{D}$ ,  $S^1(d) := S(d)$  and for  $n$  even, we set  $S^n(0) \in P^{\mathbf{I}}$
- $\mathbf{I}$  associates  $0 \in \mathbf{D}$  with  $0$ ,  $S^1$  with  $S$ ,  $P^{\mathbf{I}}$  with  $P$

## Substitutions & Assignments

### Theorem

Suppose  $t$  is closed,  $X$  is a formula of  $L$ ,  $\mathbf{M} = (\mathbf{D}, \mathbf{I})$  is a model for  $L$ . Let  $x$  be a variable and let  $\mathbf{A}$  be an assignment s.t.  $x^{\mathbf{A}} = t^{\mathbf{I}}$ . Then  $[X\{x \mapsto t\}]^{\mathbf{I}, \mathbf{A}} = X^{\mathbf{I}, \mathbf{A}}$ . More generally, if  $\mathbf{B}$  is an  $x$ -variant of  $\mathbf{A}$ , then  $[X\{x \mapsto t\}]^{\mathbf{I}, \mathbf{B}} = X^{\mathbf{I}, \mathbf{A}}$ .

### Proof

by structural induction on  $X$  □

### Theorem

Let  $\mathbf{M}$ ,  $X$ , and  $\mathbf{A}$  be as above. Let  $\sigma$  be a substitution free for  $X$ . Define  $\mathbf{B}$  by setting for each variable  $v^{\mathbf{B}} = (v\sigma)^{\mathbf{I}, \mathbf{A}}$ . Then  $X^{\mathbf{I}, \mathbf{B}} = (X\sigma)^{\mathbf{I}, \mathbf{A}}$ .

### Proof

by structural induction on  $X$  using

- for any term  $t$ :  $t^{\mathbf{I}, \mathbf{B}} = (t\sigma)^{\mathbf{I}, \mathbf{A}}$
- $\sigma$  is free for  $X$

### Theorem

if  $\mathbf{M} = (\mathbf{D}, \mathbf{I})$  is a Herbrand model for  $L$ , then for any term  $t$ :  $t^{\mathbf{I}, \mathbf{A}} = (t\mathbf{A})^{\mathbf{I}}$

### Proof

by induction on  $t$  □

### Theorem

if  $\mathbf{M} = (\mathbf{D}, \mathbf{I})$  is a Herbrand model for  $L$ , then for any formula  $X$ :  $X^{\mathbf{I}, \mathbf{A}} = (X\mathbf{A})^{\mathbf{I}}$

### Theorem

if  $A$  a formula,  $\mathbf{M} = (\mathbf{D}, \mathbf{I})$  a Herbrand model, then

- $(\forall x)A$  is true in  $\mathbf{M}$  iff  $A\{x \mapsto d\}$  is true in  $\mathbf{M}$  for every  $d \in \mathbf{D}$
- $(\exists x)A$  is true in  $\mathbf{M}$  iff  $A\{x \mapsto d\}$  is true in  $\mathbf{M}$  for some  $d \in \mathbf{D}$

## Uniform Notation

quantified formulas are grouped into two categories:

universal		existential	
$\gamma$	$\gamma(t)$	$\delta$	$\delta(t)$
$(\forall x)A$	$A\{x \mapsto t\}$	$(\exists x)A$	$A\{x \mapsto t\}$
$\neg(\exists x)A$	$\neg A\{x \mapsto t\}$	$\neg(\forall x)A$	$\neg A\{x \mapsto t\}$

we obtain

→  $\gamma \equiv (\forall y)\gamma(y)$  and  $\delta \equiv (\exists y)\delta(y)$ , provided  $y$  is new

### Theorem

suppose  $\mathbf{M} = (\mathbf{D}, \mathbf{I})$  is a **Herbrand** model for  $L$

→  $\gamma$  is true in  $\mathbf{M}$  iff  $\gamma(d)$  is true in  $\mathbf{M}$  for every  $d \in \mathbf{D}$

→  $\delta$  is true in  $\mathbf{M}$  iff  $\delta(d)$  is true in  $\mathbf{M}$  for some  $d \in \mathbf{D}$

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## Hintikka's Lemma

### Definition

first-order Hintikka set

let  $L$  be a language; a set  $\mathbf{H}$  of **sentences** is a **first-order Hintikka set** (wrt  $L$ ) if

- for any atomic formula  $A$ , not both  $A \in \mathbf{H}$  and  $\neg A \in \mathbf{H}$
- $\perp \notin \mathbf{H}$ ,  $\neg \top \notin \mathbf{H}$
- $\neg\neg Z \in \mathbf{H} \Rightarrow Z \in \mathbf{H}$
- $\alpha \in \mathbf{H} \Rightarrow \alpha_1 \in \mathbf{H}$  and  $\alpha_2 \in \mathbf{H}$
- $\beta \in \mathbf{H} \Rightarrow \beta_1 \in \mathbf{H}$  or  $\beta_2 \in \mathbf{H}$
- $\gamma \in \mathbf{H} \Rightarrow \gamma(t) \in \mathbf{H}$  for every closed term  $t$  of  $L$
- $\delta \in \mathbf{H} \Rightarrow \delta(t) \in \mathbf{H}$  for some closed term  $t$  of  $L$

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### Example

if  $L$  is defined as  $\mathbf{R} = \{P, Q\}$ ,  $\mathbf{F} = \{S\}$ ,  $\mathbf{C} = \{0\}$ ,

then

$$\{ \{ P(0), P(S^2(0)), \dots, P(S^n(0)), P(S^{n+2}(0)), \dots \} \\ \{ ((\exists x)Q(x) \vee \neg(\exists y)P(y)), (\exists x)Q(x), Q(S(S(0))), \\ \neg(\exists y)P(y), \neg P(0), \neg P(S(0)), \dots \} \}$$

are Hintikka sets wrt  $L$ , but

$$\{ ((\forall y)P(y) \wedge (\exists x)Q(x)), (\forall y)P(y), (\exists x)Q(x), Q(0) \}$$

is not

### Lemma

Hintikka's lemma

Suppose  $L$  is a language with a nonempty set of closed terms. If  $\mathbf{H}$  is a first-order Hintikka set wrt  $L$ , then  $\mathbf{H}$  is satisfiable in a Herbrand model

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### Proof

first, we construct a model  $\mathbf{M} = (\mathbf{D}, \mathbf{I})$

- $\mathbf{D}$  is the set of closed terms
- for each  $c \in \mathbf{C}$ , we set  $c^{\mathbf{I}} = c$
- for each  $f \in \mathbf{F}$  and  $t_1, \dots, t_n \in \mathbf{D}$ , we set  $f^{\mathbf{I}}(t_1, \dots, t_n) = f(t_1, \dots, t_n)$
- for each  $P \in \mathbf{R}$ , we set  $(t_1, \dots, t_n) \in P^{\mathbf{I}}$  if  $P(t_1, \dots, t_n) \in \mathbf{H}$

we obtain  $t^{\mathbf{I}} = t$  for each closed  $t$ ; hence  $\mathbf{M}$  is a **Herbrand model**; now we prove by induction on  $A$ : **if  $A \in \mathbf{H}$ , then  $A$  is true in  $\mathbf{M}$**

- **Base** suppose  $P(t_1, \dots, t_n)$  is an atomic **sentence** in  $\mathbf{H}$ ; let  $\mathbf{A}$  be an arbitrary assignment

$$(t_1, \dots, t_n) \in P^{\mathbf{I}} \quad \text{by definition of } \mathbf{M}$$

$$(t_1^{\mathbf{I}, \mathbf{A}}, \dots, t_n^{\mathbf{I}, \mathbf{A}}) \in P^{\mathbf{I}} \quad \text{for each } i: t_i^{\mathbf{I}, \mathbf{A}} = t_i^{\mathbf{I}} = t_i$$

$$[P(t_1, \dots, t_n)]^{\mathbf{I}, \mathbf{A}} = \mathbf{t}$$

- **Step** follows directly from IH and the previous theorem

□

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**Definition**

let  $L(\mathbf{R}, \mathbf{F}, \mathbf{C})$  be a language,  $\mathbf{par}$  be a countable set of constants, distinct from  $\mathbf{C}$ ;

- the members of  $\mathbf{par}$  are called **parameters**
- we write  $L^{\mathbf{par}}$  for  $L(\mathbf{R}, \mathbf{F}, \mathbf{C} \cup \mathbf{par})$

**Definition**

let  $\mathcal{C}$  be a collection of **sets of sentences** of  $L^{\mathbf{par}}$ , it is called **first-order consistency property (FCP)** if for each  $S \in \mathcal{C}$ :

- for any atomic sentence  $A$ , not both  $A \in S$  and  $\neg A \in S$
- $\perp \notin S$ ,  $\neg \top \notin S$
- $\neg \neg Z \in S \implies S \cup \{Z\} \in \mathcal{C}$
- $\alpha \in S \implies S \cup \{\alpha_1, \alpha_2\} \in \mathcal{C}$
- $\beta \in S \implies S \cup \{\beta_1\} \in \mathcal{C}$  or  $S \cup \{\beta_2\} \in \mathcal{C}$
- $\gamma \in S \implies S \cup \{\gamma(t)\} \in \mathcal{C}$  for **every closed term**  $t$  of  $L^{\mathbf{par}}$
- $\delta \in S \implies S \cup \{\delta(p)\} \in \mathcal{C}$  for **some parameter**  $p$  of  $L^{\mathbf{par}}$

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**First-Order Model Existence****Theorem**

**first-order model existence**

If  $\mathcal{C}$  is a FCP wrt  $L$ ,  $S$  is a set of sentences(!) of  $L$ , and  $S \in \mathcal{C}$ , then  $S$  is satisfiable in a Herbrand model (a Herbrand model wrt  $L^{\mathbf{par}}$ )

**Proof**

let  $\mathcal{C}$  be a FCP

- $\mathcal{C}$  **subset closed**  $S \in \mathcal{C}$  implies: for all  $S_0 \subseteq S$ ,  $S_0 \in \mathcal{C}$
- $\mathcal{C}$  **of finite character**  $S \in \mathcal{C}$  iff for any finite  $S' \subseteq S$ ,  $S' \in \mathcal{C}$

**proof plan**

- show the existence of an extension  $\mathcal{C}^*$  of  $\mathcal{C}$ , such that  $\mathcal{C}^*$  is a FCP of finite character
- define a suitable extension  $\mathbf{H}$  of  $S$ , such that  $\mathbf{H}$  is a (first-order) Hintikka set
- apply Hintikka's lemma

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**Lemma**

let  $\mathcal{C}$  be a FCP, then  $\mathcal{C}$  is extendable to a FCP  $\mathcal{C}^+$ , such that  $\mathcal{C}^+$  is subset closed

**Proof**

let  $\mathcal{C}^+$  consist of all subsets of members of  $\mathcal{C}$ , then the lemma follows easily  $\square$

moreover let  $\mathcal{C}^*$  be an extension of  $\mathcal{C}$ , we would have to show that  $\mathcal{C}^*$  admits:  $\delta \in S \implies S \cup \{\delta(p)\} \in \mathcal{C}^*$  for **some parameter**  $p$  of  $L^{\mathbf{par}}$

- **problem**: does not work!
- **solution**: change the definition of FCP

**Definition**

**alternate first-order consistency property**

let

- $\delta \in S \implies S \cup \{\delta(p)\} \in \mathcal{C}$  for **any** parameter  $p$  that is **new to**  $S$

replace the last rule in the definition of a FCP

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**Definition**

**parameter substitution**

a parameter substitution  $\pi$  is a mapping  $\pi: \mathbf{par} \rightarrow \mathbf{par}$ ,  $\pi$  is extended to a substitution on formulas (of  $L^{\mathbf{par}}$ ) and sets of formulas

**Lemma**

let  $\mathcal{C}$  be a FCP, closed under subsets; define  $\mathcal{C}^+$  as follows: if  $S\pi \in \mathcal{C}$  for **some** parameter substitution, then  $S \in \mathcal{C}^+$ ; we obtain

- $\mathcal{C}^+$  extends  $\mathcal{C}$
- $\mathcal{C}^+$  is closed under subsets
- $\mathcal{C}^+$  is an alternate FCP

**Proof**

we clarify the first point: we have to show that  $S \in \mathcal{C}$  implies  $S \in \mathcal{C}^+$

- $S \in \mathcal{C}^+$  if  $S\pi \in \mathcal{C}$  for some parameter substitution
- let  $\pi$  be the identity substitution

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**Lemma**

let  $\mathcal{C}^+$  be an alternate FCP that is subset closed, then

- ➔  $\mathcal{C}^+$  is extendable to a  $\mathcal{C}^*$  of finite character
- ➔  $\mathcal{C}^*$  is subset closed

**Proof**

let  $\mathcal{C}^*$  consist of those sets, all of whose finite subsets are in  $\mathcal{C}$ ; the lemma follows easily  $\square$

let  $X_1, X_2, \dots$  be an enumeration of all first-order sentences

$$S_1 := S$$

$$S_{n+1} := \begin{cases} S_n & \text{if } S_n \cup \{X_n\} \notin \mathcal{C}^* \\ S_n \cup \{X_n\} & \text{if } S_n \cup \{X_n\} \in \mathcal{C}^* \text{ and } X_n \text{ is not } \delta \\ S_n \cup \{X_n\} \cup \{\delta(p)\} & \text{if } S_n \cup \{X_n\} \in \mathcal{C}^*, X_n \text{ is } \delta, \\ & p \text{ a new parameter} \end{cases}$$

**NB:** each  $S_i$  leaves infinitely many parameters unused

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**Summary**

- ➔ substitutions and assignments
- ➔ Herbrand models
- ➔ uniform notation for first-order formulas
- ➔ Hintikka's lemma
- ➔ first-order model existence

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finally set

$$\mathbf{H} := \bigcup_i S_i$$

by construction  $S \in \mathbf{H}$ , furthermore

- ➔  $\mathbf{H} \in \mathcal{C}^*$ : follows from the fact that  $\mathcal{C}^*$  is of finite character
- ➔  $\mathbf{H}$  is maximal: follows by definition
- ➔  $\mathbf{H}$  is a first-order Hintikka set: we consider two cases:
  - ➔ suppose  $\gamma \in \mathbf{H}$ 
    - $\mathbf{H} \cup \{\gamma(t)\} \in \mathcal{C}^*$  for every closed term  $t$
    - $\mathbf{H} \cup \{\gamma(t)\} = \mathbf{H}$
  - ➔ suppose  $\delta \in \mathbf{H}$ 
    - $\delta \in \mathbf{H}$  implies  $\delta(p) \in \mathbf{H}$  for some parameter  $p$  (by definition)
    - hence  $\delta(t) \in \mathbf{H}$  for some term  $t$  in  $L^{\text{par}}$
- ➔  $\mathbf{H}$  is satisfiable by Hintikka's lemma
- ➔ note that  $\mathbf{H}$  is satisfiable by a Herbrand model wrt  $L^{\text{par}}$   $\square$

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