University of Innsbruck
Institute of Computer Science
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Solutions

WS 2006/07
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Final Exam
Time: 90 minutes

Automated Theorem Proving in Isabelle/HOL
LVA 703861

1 Type Inference in Simply-Typed Lambda Calculus **10 points**

Give a most general type for the following λ-term. Show the full typing derivation.

$$\lambda x\, y.\, x\,(x\,y)$$

Solution:

$\lambda x\, y.\, x\,(x\,y)$ has the most general type $(\alpha \Rightarrow \alpha) \Rightarrow \alpha \Rightarrow \alpha$. The typing derivation is given by

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash x :: \alpha \Rightarrow \alpha \qquad \dfrac{\Gamma \vdash x :: \alpha \Rightarrow \alpha \qquad \Gamma \vdash y :: \alpha}{\Gamma \vdash x\,y :: \alpha}}{\Gamma \vdash x\,(x\,y) :: \alpha}}{[x \leftarrow \alpha \Rightarrow \alpha] \vdash \lambda y.\, x\,(x\,y) :: \alpha \Rightarrow \alpha}}{[] \vdash \lambda x\, y.\, x\,(x\,y) :: (\alpha \Rightarrow \alpha) \Rightarrow \alpha \Rightarrow \alpha}$$

where $\Gamma = [x \leftarrow \alpha \Rightarrow \alpha, y \leftarrow \alpha]$.

2 Natural Deduction **10 points**

Give a complete set of natural-deduction style introduction and elimination rules (without using conjunction/disjunction) for *exclusive disjunction* (xor, $\oplus$), defined by the following truth table:

P	Q	$P \oplus Q$
False	False	False
False	True	True
True	False	True
True	True	False

Solution:

We have two introduction rules, xorI1 and xorI2, and one elimination rule, xorE. We state the latter with a general conclusion R, i.e. in the same format as an elimination rule in Isabelle that should be applicable to arbitrary goals:

$$\frac{\neg P \qquad Q}{P \oplus Q}\,\text{xorI1}$$

$$\frac{P \qquad \neg Q}{P \oplus Q}\,\text{xorI2}$$

$$\frac{P \oplus Q \qquad [\![\neg P; Q]\!] \Longrightarrow R \qquad [\![P; \neg Q]\!] \Longrightarrow R}{R}\,\text{xorE}$$

3 Modeling/Formalization **10 points**

Give a formalization of the following puzzle in (a suitable subset of) higher-order logic. Explain your formalization. (You do *not* need to solve the puzzle.)

> Human observers in this exclusive club on Ganymede can't distinguish Martians from Venusians, males from females, except for the fact that Venusian women and Martian men always tell the truth and Venusian men and Martian women always lie. (Everyone is either Martian or Venusian, and either male or female.) A says "B is from Venus." B says "A is from Mars." A says "B is male." B says "A is female." Who's what (sex and planet of origin)?

Solution:

We formalize the puzzle in propositional logic. Let AV be a Boolean variable that is true iff A is from Venus. Let AW be a Boolean variable that is true iff A is a woman. Likewise, let BV denote that B is from Venus, and let BW denote that B is a woman.

Since everyone is either Martian or Venusian, and either male or female, A is from Mars if and only if A is not from Venus, and A is male if and only if A is not female. Likewise for B.

A's statements are true if A is a Venusian woman (i.e. $AV \wedge AW$), or if A is a Martian man (i.e. $\neg AV \wedge \neg AW$). Hence A's statements are true iff $AV = AW$. Likewise, B's statements are true iff $BV = BW$.

The four statements made by A and B and their respective truth are now given by the following four formulas:

- $BV = (AV = AW)$
- $\neg AV = (BV = BW)$
- $\neg BW = (AV = AW)$
- $AW = (BV = BW)$

Solving the puzzle amounts to finding an assignment for the Boolean variables AV, AW, BV, BW that satisfies these formulas.

Note: A formalization in first-order logic is possible as well.

4 Inductive Definitions **15 points**

Give a formalization of the following puzzle in (a suitable subset of) higher-order logic. Explain your formalization. (You do *not* need to solve the puzzle.)

> There are 3 missionaries, 3 cannibals, and a boat on the west bank of a river. All wish to cross, but the boat holds at most 2 people. If the cannibals ever outnumber the missionaries on either bank of the river the outnumbered missionaries will be eaten. Can they all safely cross the river? (The boat cannot cross empty.)

Hint: Find a suitable notion of "state". Characterize the reachable states inductively.

Solution:

We define a state to be a triple (b, m, c), where the first component is either *east* or *west* (indicating the position of the boat), the second component gives the number of missionaries on the west bank, and the third component gives the number of cannibals on the west bank. (Since we are only interested in states where nobody got eaten, the number of missionaries on the east bank is always $3 - m$. The number of cannibals on the east bank is $3 - c$.)

We introduce a predicate R on states to characterize those states that are reachable.

The initial state is reachable:

$$R\,(west, 3, 3).$$

The boat can take one or two people from the west bank to the east bank, provided that the missionaries will not be outnumbered on either bank afterwards (mb and cb are non-negative integers, indicating the number of missionaries and cannibals, respectively, on the boat):

$$[\![R\,(west, m, c);\; mb \le m;\; cb \le c; mb + cb \in \{1, 2\};\; c - cb \le m - mb;\; 3 - (c - cb) \le 3 - (m - mb)]\!]$$

$$\Longrightarrow R\,(east, m - mb, c - cb).$$

Likewise, the boat can take one or two people from the east bank to the west bank under similar conditions:

$$[\![R\,(east, m, c);\; mb \le 3{-}m;\; cb \le 3{-}c; mb{+}cb \in \{1, 2\};\; c{+}cb \le m{+}mb;\; 3{-}(c{+}cb) \le 3{-}(m{+}mb)]\!]$$

$$\Longrightarrow R\,(west, m + mb, c + cb).$$

The missionaries and cannibals can all safely cross the river if and only if

$$R\,(east, 0, 0)$$

is provable from these clauses.

5 Isabelle/Isar **15 points**

Give an Isar proof of the following lemma. The general proof structure should resemble that of a detailed informal (not necessarily natural deduction) proof.

```
lemma "(∃x. ∀y. P x y) ⟹ (∀y. ∃x. P x y)"
proof
  assume "∃x. ∀y. P x y"
  then obtain x where X: "∀y. P x y" ..
  fix y
  from X have "P x y" ..
  thus "∃x. P x y" ..
qed
```

You know my methods. Apply them!
Sherlock Holmes