Advanced Topics in Term Rewriting
LVA 703610

`http://cl-informatik.uibk.ac.at/teaching/ws06/attr/`

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office hours: Tuesday, 16:00–18:00 (3M09)

Schedule

week 1	October 5	week 7	November 30
week 2	October 12	week 8	December 7
week 3	October 19	week 9	December 14
week 4	November 9	week 10	January 11
week 5	no lecture	week 11	January 18
week 6	November 23	first exam	January 25

Content

- Termination
  - Semantic Labelling
  - Dependency Pairs
- Complexity
  - Computational Complexity (?)
  - Derivational Complexity
- Proof Techniques
  - Kruskal's Theorem
  - Tree Automata (?)

Term Rewriting

- signature 0 constant S unary $+$ $\times$ binary
- rewrite rules

$$\begin{aligned} 0 + x &\to x \\ \mathsf{S}(x) + y &\to \mathsf{S}(x + y) \\ 0 \times x &\to 0 \\ \mathsf{S}(x) \times y &\to x \times y + y \end{aligned}$$

TRS

- rewriting

$$\begin{aligned} & \mathsf{S}(0) + \mathsf{S}(\mathsf{S}(0) \times \mathsf{S}(\mathsf{S}(0))) \\ \to\ & \mathsf{S}(0) + \mathsf{S}(0 \times \mathsf{S}(\mathsf{S}(0)) + \mathsf{S}(\mathsf{S}(0))) \\ \to\ & \mathsf{S}(0) + \mathsf{S}(0 + \mathsf{S}(\mathsf{S}(0))) \\ \to\ & \mathsf{S}(0) + \mathsf{S}(\mathsf{S}(\mathsf{S}(0))) \\ \to\ & \mathsf{S}(\mathsf{S}(\mathsf{S}(\mathsf{S}(0)))) \end{aligned}$$

normal form

- **signature** $0, 1, \ldots 9$ constants $+, :$ binary
- **rewrite rules**

$$
\begin{array}{llll}
0+0 \to 0 & 1+0 \to 1 & \cdots & 9+0 \to 9 \\
0+1 \to 1 & 1+1 \to 2 & \cdots & 9+1 \to 1:0 \\
0+2 \to 2 & 1+2 \to 3 & \cdots & 9+2 \to 1:1 \\
0+3 \to 3 & 1+3 \to 4 & \cdots & 9+3 \to 1:2 \\
0+4 \to 4 & 1+4 \to 5 & \cdots & 9+4 \to 1:3 \\
0+5 \to 5 & 1+5 \to 6 & \cdots & 9+5 \to 1:4 \\
0+6 \to 6 & 1+6 \to 7 & \cdots & 9+6 \to 1:5 \\
0+7 \to 7 & 1+7 \to 8 & \cdots & 9+7 \to 1:6 \\
0+8 \to 8 & 1+8 \to 9 & \cdots & 9+8 \to 1:7 \\
0+9 \to 9 & 1+9 \to 1:0 & \cdots & 9+9 \to 1:8
\end{array}
$$

$$
\begin{array}{ll}
x + (y : z) \to y : (x + z) & 0 : x \to x \\
(x : y) + z \to x : (y + z) & x : (y : z) \to (x + y) : z
\end{array}
$$

- **rewriting**

$$
\begin{array}{lll}
(2:3)+(7:7) & \to 7:(2:3)+7 & \\
\to 7:(2:(3+7)) & \to 7:(2:(1:0)) & \to 7:((2+1):0) \\
\to 7:(3:0) & \to (7+3):0 & \to (1:0):0
\end{array}
$$

- **signature** $0, \mathsf{fib}$ constants S unary $\mathsf{f}, +, :$ binary
- **rules**

$$
\begin{array}{rclrcl}
0 + y & \to & y & \mathsf{fib} & \to & \mathsf{f}(\mathsf{S}(0), \mathsf{S}(0)) \\
\mathsf{S}(x) + y & \to & \mathsf{S}(x + y) & \mathsf{f}(x, y) & \to & x : \mathsf{f}(y, x + y)
\end{array}
$$

- **rewriting**

$$
\begin{array}{lll}
\mathsf{fib} & \to & \mathsf{f}(\mathsf{S}(0), \mathsf{S}(0)) \\
& \to & \mathsf{S}(0) : \mathsf{f}(\mathsf{S}(0), \mathsf{S}(0) + \mathsf{S}(0)) \\
& \to & \mathsf{S}(0) : \mathsf{f}(\mathsf{S}(0), \mathsf{S}(0 + \mathsf{S}(0))) \\
& \to & \mathsf{S}(0) : \mathsf{f}(\mathsf{S}(0), \mathsf{S}(\mathsf{S}(0))) \\
& \to & \mathsf{S}(0) : \mathsf{S}(0) : \mathsf{f}(\mathsf{S}(\mathsf{S}(0)), \mathsf{S}(0) + \mathsf{S}(\mathsf{S}(0))) \\
& \to^+ & \mathsf{S}(0) : \mathsf{S}(0) : \mathsf{f}(\mathsf{S}(\mathsf{S}(0)), \mathsf{S}(\mathsf{S}(\mathsf{S}(0)))) \\
& \to^+ & \mathsf{S}(0) : \mathsf{S}(0) : \mathsf{S}^2(0) : \mathsf{f}(\mathsf{S}^3(0), \mathsf{S}^5(0)) \\
& \to^+ & \mathsf{S}(0) : \mathsf{S}(0) : \mathsf{S}^2(0) : \mathsf{S}^3(0) : \mathsf{f}(\mathsf{S}^5(0), \mathsf{S}^8(0))
\end{array}
$$

infinite computation

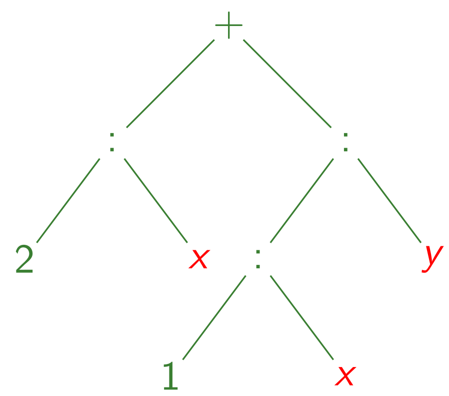
Terms

- signature $\mathcal{F}$ function symbols with arities
- variables $\mathcal{V}$ $\mathcal{F} \cap \mathcal{V} = \varnothing$ infinitely many
- terms $\mathcal{T}(\mathcal{F}, \mathcal{V})$
- ground terms $\mathcal{T}(\mathcal{F})$

Operations

- $\mathsf{Var}(t)$ x y
- $\mathsf{FS}(t)$ 1 2 $:$ $+$
- $\mathsf{root}(t)$ $+$

$(2 : x) + ((1 : x) : y)$

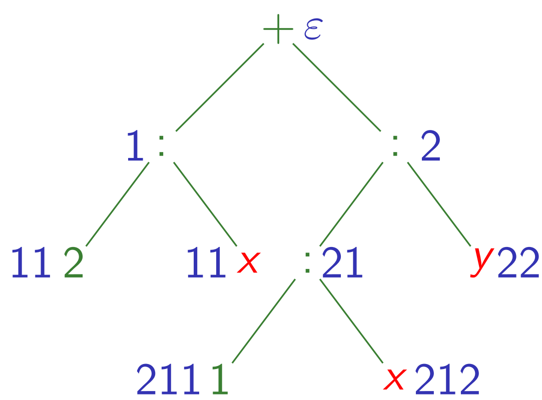


Subterms and Positions

Definition

- $s \trianglelefteq t$ s is subterm of t
- $t|_p$ take subterm of t at position p
- $t[s]_p$ replace subterm in t at position p by s
- $\mathcal{P}\mathsf{os}(t) = \mathcal{P}\mathsf{os}_{\mathcal{F}}(t) \cup \mathcal{P}\mathsf{os}_{\mathcal{V}}(t)$
- $p \leqslant q$ above
- $p \parallel q$ parallel

$(2 : x) + ((1 : x) : y)$



Substitutions

- substitution is mapping $\sigma\colon \mathcal{V} \to \mathcal{T}(\mathcal{F},\mathcal{V})$ such that

$$\mathcal{D}\mathrm{om}(\sigma) = \underbrace{\{x \in \mathcal{V} \mid \sigma(x) \neq x\}}_{\text{domain}}$$

  is finite

- application of substitution σ to term t:

$$t\sigma = \begin{cases} \sigma(t) & \text{if } t \text{ is variable} \\ f(t_1\sigma, \ldots, t_n\sigma) & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

- empty substitution ε $\quad (\mathcal{D}\mathrm{om}(\varepsilon) = \varnothing)$

Term Rewrite Systems

- rewrite rule $(l \to r)$ is pair of terms l, r such that
  1. $l \notin \mathcal{V}$
  2. $\mathrm{Var}(r) \subseteq \mathrm{Var}(l)$
- term rewrite system (TRS) is pair $(\mathcal{F}, \mathcal{R})$
  1. $\mathcal{F}$ signature
  2. $\mathcal{R}$ set of rewrite rules between terms in $\mathcal{T}(\mathcal{F},\mathcal{V})$
- binary relation $\to_{\mathcal{R}}$ on $\mathcal{T}(\mathcal{F},\mathcal{V})$ for every TRS $(\mathcal{F}, \mathcal{R})$:

$$s \to_{\mathcal{R}} t \quad \Longleftrightarrow \quad \begin{array}{l} \exists\ p \in \mathcal{P}\mathrm{os}(s) \\ \exists\ l \to r \in \mathcal{R} \\ \exists \text{ substitution } \sigma \end{array} \quad \text{with} \quad \begin{array}{rcl} s|_p & = & l\sigma \quad \text{redex} \\ t & = & s[r\sigma]_p \end{array}$$

Termination

Definition

TRS is terminating if there are no infinite rewrite sequences

Theorem

TRS $\mathcal{R}$ is terminating iff $\exists$ well-founded order $>$ on terms such that

$$s \to_{\mathcal{R}} t \quad \Longrightarrow \quad s > t$$

inconvenient to check all rewrite steps

➡ but: sometimes induction over the term structure, together with a well-founded relation cannot be avoided

Theorem

TRS $\mathcal{R}$ is terminating iff $\exists$ well-founded order $>$ on terms such that

1. $l \to r \in \mathcal{R} \quad \Longrightarrow \quad l > r$
2. $>$ is closed under contexts $(s > t \;\Rightarrow\; u[s]_p > u[t]_p)$
3. $>$ is closed under substitutions $(s > t \;\Rightarrow\; s\sigma > t\sigma)$

Definition

binary relation $>$ on terms is reduction order if

1. closed under contexts
2. closed under substitutions
3. proper order (irreflexive and transitive)
4. well-founded

Definition

TRS $\mathcal{R}$ and $>$ are compatible if $l > r$ for all $l \to r \in \mathcal{R}$

Theorem

TRS $\mathcal{R}$ is terminating iff compatible with reduction order

Question

how to construct reduction orders ?

1. use algebras (semantic approach)
2. use induction (syntactic approach)

Definition

- precedence is proper order $>$ on $\mathcal{F}$
- relation $>_{\mathsf{lpo}}$ (lexicographic path order) on terms:
  $s >_{\mathsf{lpo}} t$ if $s = f(s_1, \ldots, s_n)$ and either
  1. $\exists\, i\ s_i >_{\mathsf{lpo}} t$ or $s_i = t$,
  2. $t = g(t_1, \ldots, t_m)$ and $f > g$ and $\forall\, j\ s >_{\mathsf{lpo}} t_j$, or
  3. $t = f(t_1, \ldots, t_n)$ and $\exists\, i$

$$\forall\, j \in [1, i-1]\ s_j = t_j \qquad s_i >_{\mathsf{lpo}} t_i \qquad \forall\, j > i\ s >_{\mathsf{lpo}} t_j$$

Theorem

$>_{\mathsf{lpo}}$ is reduction order if $>$ is well-founded

$$\begin{aligned} x + 0 &\to x \\ x + \mathsf{S}(y) &\to \mathsf{S}(x + y) \\ x \times 0 &\to 0 \\ x \times \mathsf{S}(y) &\to x \times y + x \end{aligned} \qquad \times > + > \mathsf{S}$$

Theorem

- if $> \subseteq \sqsupset$ then $>_{\mathsf{lpo}} \; > \subseteq \; >_{\mathsf{lpo}} \sqsupset$ (incrementality)
- if $>$ is total then $>_{\mathsf{lpo}}$ is total on ground terms (well-order)
- following two problems are decidable:
  1. instance: terms s, t $>$
     question: $s >_{\mathsf{lpo}} t$?
  2. instance: terms s, t
     question: $\exists$ precedence $>$ such that $s >_{\mathsf{lpo}} t$?

$$
\begin{aligned}
\mathsf{ack}(0,0) &\to 0 \\
\mathsf{ack}(0,\mathsf{S}(y)) &\to \mathsf{S}(\mathsf{S}(\mathsf{ack}(0,y))) \\
\mathsf{ack}(\mathsf{S}(x),0) &\to \mathsf{S}(0) \\
\mathsf{ack}(\mathsf{S}(x),\mathsf{S}(y)) &\to \mathsf{ack}(x,\mathsf{ack}(\mathsf{S}(x),y))
\end{aligned}
$$

$\mathsf{ack} > \mathsf{S}$

Definition

- precedence is proper order $>$ on $\mathcal{F}$
- relation $>_{\mathsf{mpo}}$ (multiset path order) on terms:
  $s >_{\mathsf{mpo}} t$ if $s = f(s_1, \ldots, s_n)$ and either
  1. $\exists\, i \; s_i >_{\mathsf{mpo}} t$ or $s_i = t$
  2. $t = g(t_1, \ldots, t_m)$ and $f > g$ and $\forall\, j \; s >_{\mathsf{mpo}} t_j$
  3. $t = f(t_1, \ldots, t_n)$ and $\{s_1, \ldots, s_n\} >_{\mathsf{mpo}}^{\mathsf{mul}} \{t_1, \ldots, t_n\}$

$$
M >_{\mathsf{mpo}}^{\mathsf{mul}} N \iff \overbrace{M - N}^{\text{multiset difference}} \neq \varnothing \;\wedge\; \forall\, t \in N - M \; \exists\, s \in M - N \;\; s >_{\mathsf{mpo}} t
$$

Theorem

$>_{\mathsf{mpo}}$ is reduction order if $>$ is well-founded

Definition

- weight function (w, w_0) consists of mapping $\mathsf{w}\colon \mathcal{F} \to \mathbb{N}$ and constant $w_0 > 0$ such that $\mathsf{w}(c) \geq w_0$ for all constants $c \in \mathcal{F}$
- weight of term t is

$$\mathsf{w}(t) = w_0 \cdot \Big(\sum_{x \in \mathsf{Var}(t)} |t|_x \Big) + \sum_{f \in \mathsf{FS}(t)} \mathsf{w}(f) \cdot |t|_f$$

- weight function (w, w_0) is admissible for precedence $>$ if

$$f > g \text{ for all } g \in \mathcal{F} \setminus \{f\}$$

whenever f is unary function symbol in $\mathcal{F}$ with $w(f) = 0$

$$\mathsf{w}(+) = \mathsf{w}(\mathsf{S}) = 0 \quad \mathsf{w}(0) = 1 \qquad \mathsf{S} > \circ > 0$$

Definition

- precedence is proper order $>$ on $\mathcal{F}$
- admissible weight function (w, w_0)
- relation $>_{\mathsf{kbo}}$ (Knuth-Bendix order) on terms:
  $s >_{\mathsf{kbo}} t$ if $|s|_x \geq |t|_x$ for all $x \in \mathcal{V}$ and either

  1. $w(s) > w(t)$,
  2. $w(s) = w(t)$ and either
     1. $\exists\, n > 0\ \exists\, x \in \mathcal{V}\ s = f^n(x)$ and $t = x$
     2. $s = f(s_1, \dots, s_n)$ and $t = f(t_1, \dots, t_n)$ and $\exists\, i$

        $$\forall j < i\ s_j = t_j \qquad s_i >_{\mathsf{kbo}} t_i$$

     3. $s = f(s_1, \dots, s_n)$ and $t = g(t_1, \dots, t_m)$ and $f > g$

Theorem

$>_{\mathsf{kbo}}$ is reduction order if $>$ is well-founded and (w, w_0) admissible

Theorem

- if $> \subseteq \sqsupset$ and (w, w_0) admissible then $>_{\mathsf{kbo}} \; > \subseteq \; >_{\mathsf{kbo}} \; \sqsupset$ (incrementality)
- if $>$ is total then $>_{\mathsf{kbo}}$ is total on ground terms (well-order)
- following two problems are decidable:

  1. instance: terms s, t $\quad >\quad$ (w, w_0)
     question: $s >_{\mathsf{kbo}} t$?
  2. instance: terms s, t
     question: $\exists$ precedence $>$ and admissible (w, w_0) such that $s >_{\mathsf{kbo}} t$?

$$\begin{aligned} \mathsf{g}(\mathsf{g}(x)) &\rightarrow \mathsf{f}(x) \\ \mathsf{f}(\mathsf{g}(x)) &\rightarrow \mathsf{g}(\mathsf{f}(x)) \end{aligned} \qquad \mathsf{f} > \mathsf{g} \wedge \mathsf{w}(\mathsf{f}) = \mathsf{w}(\mathsf{g}) = 1$$

Definition

- well-founded monotone $\mathcal{F}$-algebra (WFMA) $(\mathcal{A}, >)$ is non-empty algebra $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f \in \mathcal{F}})$ together with well-founded order $>$ on A such that every $f_{\mathcal{A}}$ is strictly monotone in all coordinates:

$$f_{\mathcal{A}}(a_1, \ldots, a_i, \ldots, a_n) \; > \; f_{\mathcal{A}}(a_1, \ldots, b, \ldots, a_n)$$

  for all $a_1, \ldots, a_n, b \in A$ and $i \in [1, n]$ with $a_i > b$

- binary relation $>_{\mathcal{A}}$ on terms:

$$s >_{\mathcal{A}} t \iff \underbrace{[\alpha]_{\mathcal{A}}(s)} > [\alpha]_{\mathcal{A}}(t) \quad \text{for all assignments } \alpha$$

  interpretation of s in $\mathcal{A}$ under assignment α

- TRS $\mathcal{R}$ and WFMA $(\mathcal{A}, >)$ are compatible if $\mathcal{R}$ and $>_{\mathcal{A}}$ are compatible

Theorem

- $>_{\mathcal{A}}$ is reduction order for every WFMA $(\mathcal{A}, >)$
- TRS is terminating iff compatible with WFMA

Definition

TRS $\mathcal{R}$ is polynomially terminating if compatible with WFMA $(\mathcal{A}, >)$ such that

1. carrier of $\mathcal{A}$ is $\mathbb{N}$
2. $>$ is standard order on $\mathbb{N}$
3. $f_{\mathcal{A}}$ is polynomial for every f

$$
\begin{aligned}
x + 0 &\to x & 0 &:= 1 \\
x + \mathsf{S}(y) &\to \mathsf{S}(x + y) & \mathsf{S}_{\mathcal{A}} &:= \lambda x\,.\,x + 1 \\
x \times 0 &\to 0 & +_{\mathcal{A}} &:= \lambda xy\,.\,x + 2y \\
x \times \mathsf{S}(y) &\to x \times y + x & \times_{\mathcal{A}} &:= \lambda xy\,.\,(x+1)(y+1)^2
\end{aligned}
$$

History

interpretation method	Turing	1949
polynomial interpretations	Lankford	1975
	Ben Cherifa, Lescanne	1987
lexicographic path order	Schütte	1960
	Dershowitz	1982
	Kamin, Lévy	1980
Knuth-Bendix order	Knuth, Bendix	1970
	Dick, Kalmus, Martin	1990
recursive decomposition order		
	Jouannaud, Lescanne, Reinig	1982

Remark

traditional termination methods yield simple termination