Advanced Topics in Term Rewriting
LVA 703610

`http://cl-informatik.uibk.ac.at/teaching/ws06/attr/`

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office hours: Tuesday, 16:00–18:00 (3M09)

Schedule

week 1	October 5	week 7	November 30
week 2	October 12	week 8	December 7
week 3	October 19	week 9	December 14
week 4	November 9	week 10	January 11
week 5	no lecture	week 11	January 18
week 6	November 23	first exam	January 25

Content

- Termination
  - Semantic Labelling
  - Dependency Pairs
- Complexity
  - Computational Complexity (?)
  - Derivational Complexity
- Proof Techniques
  - Kruskal's Theorem
  - Tree Automata (?)

Term Rewriting

- signature: 0 constant, S unary, $+$ $\times$ binary
- rewrite rules (TRS)

$$\begin{aligned} 0 + x &\to x \\ \mathsf{S}(x) + y &\to \mathsf{S}(x + y) \\ 0 \times x &\to 0 \\ \mathsf{S}(x) \times y &\to x \times y + y \end{aligned}$$

- rewriting

$$\begin{aligned} & \mathsf{S}(0) + \mathsf{S}(\mathsf{S}(0) \times \mathsf{S}(\mathsf{S}(0))) \\ \to\ & \mathsf{S}(0) + \mathsf{S}(0 \times \mathsf{S}(\mathsf{S}(0)) + \mathsf{S}(\mathsf{S}(0))) \\ \to\ & \mathsf{S}(0) + \mathsf{S}(0 + \mathsf{S}(\mathsf{S}(0))) \\ \to\ & \mathsf{S}(0) + \mathsf{S}(\mathsf{S}(\mathsf{S}(0))) \\ \to\ & \mathsf{S}(\mathsf{S}(\mathsf{S}(\mathsf{S}(0)))) \end{aligned}$$

normal form

- ➡ signature $0, 1, \ldots 9$ constants $+, :$ binary
- ➡ rewrite rules

$0+0 \to 0$	$1+0 \to 1$	$\cdots$	$9+0 \to 9$
$0+1 \to 1$	$1+1 \to 2$	$\cdots$	$9+1 \to 1:0$
$0+2 \to 2$	$1+2 \to 3$	$\cdots$	$9+2 \to 1:1$
$0+3 \to 3$	$1+3 \to 4$	$\cdots$	$9+3 \to 1:2$
$0+4 \to 4$	$1+4 \to 5$	$\cdots$	$9+4 \to 1:3$
$0+5 \to 5$	$1+5 \to 6$	$\cdots$	$9+5 \to 1:4$
$0+6 \to 6$	$1+6 \to 7$	$\cdots$	$9+6 \to 1:5$
$0+7 \to 7$	$1+7 \to 8$	$\cdots$	$9+7 \to 1:6$
$0+8 \to 8$	$1+8 \to 9$	$\cdots$	$9+8 \to 1:7$
$0+9 \to 9$	$1+9 \to 1:0$	$\cdots$	$9+9 \to 1:8$

$$x + (y : z) \to y : (x + z) \qquad 0 : x \to x$$
$$(x : y) + z \to x : (y + z) \qquad x : (y : z) \to (x + y) : z$$

- ➡ rewriting

$$(2:3)+(7:7) \to 7:(2:3)+7$$
$$\to 7:(2:(3+7)) \to 7:(2:(1:0)) \to 7:((2+1):0)$$
$$\to 7:(3:0) \to (7+3):0 \to (1:0):0$$

- ➡ signature $0, \mathsf{fib}$ constants S unary $\mathsf{f}, +, :$ binary
- ➡ rules

$$0 + y \to y \qquad \mathsf{fib} \to \mathsf{f}(\mathsf{S}(0), \mathsf{S}(0))$$
$$\mathsf{S}(x) + y \to \mathsf{S}(x + y) \qquad \mathsf{f}(x, y) \to x : \mathsf{f}(y, x + y)$$

- ➡ rewriting

$$\begin{aligned}
\mathsf{fib} &\to \mathsf{f}(\mathsf{S}(0), \mathsf{S}(0)) \\
&\to \mathsf{S}(0) : \mathsf{f}(\mathsf{S}(0), \mathsf{S}(0) + \mathsf{S}(0)) \\
&\to \mathsf{S}(0) : \mathsf{f}(\mathsf{S}(0), \mathsf{S}(0 + \mathsf{S}(0))) \\
&\to \mathsf{S}(0) : \mathsf{f}(\mathsf{S}(0), \mathsf{S}(\mathsf{S}(0))) \\
&\to \mathsf{S}(0) : \mathsf{S}(0) : \mathsf{f}(\mathsf{S}(\mathsf{S}(0)), \mathsf{S}(0) + \mathsf{S}(\mathsf{S}(0))) \\
&\to^+ \mathsf{S}(0) : \mathsf{S}(0) : \mathsf{f}(\mathsf{S}(\mathsf{S}(0)), \mathsf{S}(\mathsf{S}(\mathsf{S}(0)))) \\
&\to^+ \mathsf{S}(0) : \mathsf{S}(0) : \mathsf{S}^2(0) : \mathsf{f}(\mathsf{S}^3(0), \mathsf{S}^5(0)) \\
&\to^+ \mathsf{S}(0) : \mathsf{S}(0) : \mathsf{S}^2(0) : \mathsf{S}^3(0) : \mathsf{f}(\mathsf{S}^5(0), \mathsf{S}^8(0))
\end{aligned}$$

infinite computation

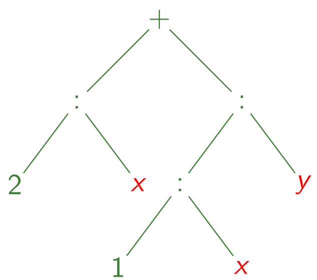
Terms

- ➡ signature $\mathcal{F}$ function symbols with arities
- ➡ variables $\mathcal{V}$ $\mathcal{F} \cap \mathcal{V} = \varnothing$ infinitely many
- ➡ terms $\mathcal{T}(\mathcal{F}, \mathcal{V})$
- ➡ ground terms $\mathcal{T}(\mathcal{F})$

Operations

- ➡ $\mathsf{Var}(t)$ x y
- ➡ $\mathsf{FS}(t)$ 1 2 $:$ $+$
- ➡ $\mathsf{root}(t)$ $+$

$(2 : x) + ((1 : x) : y)$

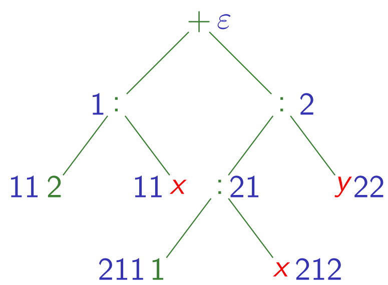


Subterms and Positions

Definition

- ➡ $s \trianglelefteq t$ s is subterm of t
- ➡ $t|_p$ take subterm of t at position p
- ➡ $t[s]_p$ replace subterm in t at position p by s
- ➡ $\mathcal{P}\mathsf{os}(t) = \mathcal{P}\mathsf{os}_{\mathcal{F}}(t) \cup \mathcal{P}\mathsf{os}_{\mathcal{V}}(t)$
- ➡ $p \leqslant q$ above
- ➡ $p \parallel q$ parallel

$(2 : x) + ((1 : x) : y)$



Substitutions

➡ substitution is mapping $\sigma\colon \mathcal{V} \to \mathcal{T}(\mathcal{F},\mathcal{V})$ such that

$$\mathcal{D}om(\sigma) = \underbrace{\{x \in \mathcal{V} \mid \sigma(x) \neq x\}}_{\text{domain}}$$

is finite

➡ application of substitution σ to term t:

$$t\sigma = \begin{cases} \sigma(t) & \text{if } t \text{ is variable} \\ f(t_1\sigma,\dots,t_n\sigma) & \text{if } t = f(t_1,\dots,t_n) \end{cases}$$

➡ empty substitution ε $\quad (\mathcal{D}om(\varepsilon) = \varnothing)$

Term Rewrite Systems

➡ rewrite rule ($l \to r$) is pair of terms l, r such that

1. $l \notin \mathcal{V}$
2. $\mathsf{Var}(r) \subseteq \mathsf{Var}(l)$

➡ term rewrite system (TRS) is pair $(\mathcal{F},\mathcal{R})$

1. $\mathcal{F}$ signature
2. $\mathcal{R}$ set of rewrite rules between terms in $\mathcal{T}(\mathcal{F},\mathcal{V})$

➡ binary relation $\to_{\mathcal{R}}$ on $\mathcal{T}(\mathcal{F},\mathcal{V})$ for every TRS $(\mathcal{F},\mathcal{R})$:

$$s \to_{\mathcal{R}} t \iff \begin{array}{l} \exists\ p \in \mathcal{P}os(s) \\ \exists\ l \to r \in \mathcal{R} \\ \exists \text{ substitution } \sigma \end{array} \quad \text{with} \quad \begin{array}{rcl} s|_p & = & l\sigma \quad \text{redex} \\ t & = & s[r\sigma]_p \end{array}$$

Termination

Definition

TRS is terminating if there are no infinite rewrite sequences

Theorem

TRS $\mathcal{R}$ is terminating iff $\exists$ well-founded order $>$ on terms such that

$$s \to_{\mathcal{R}} t \implies s > t$$

inconvenient to check all rewrite steps

➡ but: sometimes induction over the term structure, together with a well-founded relation cannot be avoided

Theorem

TRS $\mathcal{R}$ is terminating iff $\exists$ well-founded order $>$ on terms such that

1. $l \to r \in \mathcal{R} \implies l > r$
2. $>$ is closed under contexts $\quad (s > t \Rightarrow u[s]_p > u[t]_p)$
3. $>$ is closed under substitutions $\quad (s > t \Rightarrow s\sigma > t\sigma)$

Definition

binary relation $>$ on terms is reduction order if

1. closed under contexts
2. closed under substitutions
3. proper order (irreflexive and transitive)
4. well-founded

Definition

TRS $\mathcal{R}$ and $>$ are compatible if $l > r$ for all $l \to r \in \mathcal{R}$

Theorem

TRS $\mathcal{R}$ is terminating iff compatible with reduction order

Question

how to construct reduction orders ?

1. use algebras (semantic approach)
2. use induction (syntactic approach)

Definition

- precedence is proper order $>$ on $\mathcal{F}$
- relation $>_{\mathsf{lpo}}$ (lexicographic path order) on terms:
  $s >_{\mathsf{lpo}} t$ if $s = f(s_1, \dots, s_n)$ and either
  1. $\exists\, i\ s_i >_{\mathsf{lpo}} t$ or $s_i = t$,
  2. $t = g(t_1, \dots, t_m)$ and $f > g$ and $\forall\, j\ s >_{\mathsf{lpo}} t_j$, or
  3. $t = f(t_1, \dots, t_n)$ and $\exists\, i$

$$\forall\, j \in [1, i-1]\ s_j = t_j \qquad s_i >_{\mathsf{lpo}} t_i \qquad \forall\, j > i\ s >_{\mathsf{lpo}} t_j$$

Theorem

$>_{\mathsf{lpo}}$ is reduction order if $>$ is well-founded

$$\begin{aligned} x + 0 &\to x \\ x + \mathsf{S}(y) &\to \mathsf{S}(x + y) \\ x \times 0 &\to 0 \\ x \times \mathsf{S}(y) &\to x \times y + x \end{aligned} \qquad \times > + > \mathsf{S}$$

Theorem

- if $> \subseteq \sqsupset$ then $>_{\mathsf{lpo}} \subseteq >_{\mathsf{lpo}}{}_{\sqsupset}$ (incrementality)
- if $>$ is total then $>_{\mathsf{lpo}}$ is total on ground terms (well-order)
- following two problems are decidable:
  1. instance: terms s, t $>$
     question: $s >_{\mathsf{lpo}} t$?
  2. instance: terms s, t
     question: $\exists$ precedence $>$ such that $s >_{\mathsf{lpo}} t$?

$$\begin{aligned} \mathsf{ack}(0,0) &\to 0 \\ \mathsf{ack}(0,\mathsf{S}(y)) &\to \mathsf{S}(\mathsf{S}(\mathsf{ack}(0,y))) \\ \mathsf{ack}(\mathsf{S}(x),0) &\to \mathsf{S}(0) \\ \mathsf{ack}(\mathsf{S}(x),\mathsf{S}(y)) &\to \mathsf{ack}(x,\mathsf{ack}(\mathsf{S}(x),y)) \end{aligned} \qquad \mathsf{ack} > \mathsf{S}$$

Definition

- precedence is proper order $>$ on $\mathcal{F}$
- relation $>_{\mathsf{mpo}}$ (multiset path order) on terms:
  $s >_{\mathsf{mpo}} t$ if $s = f(s_1, \dots, s_n)$ and either
  1. $\exists\, i\ s_i >_{\mathsf{mpo}} t$ or $s_i = t$
  2. $t = g(t_1, \dots, t_m)$ and $f > g$ and $\forall\, j\ s >_{\mathsf{mpo}} t_j$
  3. $t = f(t_1, \dots, t_n)$ and $\{s_1, \dots, s_n\} >_{\mathsf{mpo}}^{\mathsf{mul}} \{t_1, \dots, t_n\}$

$$M >_{\mathsf{mpo}}^{\mathsf{mul}} N \iff \overbrace{M - N}^{\text{multiset difference}} \neq \varnothing \ \wedge\ \forall\, t \in N - M\ \exists\, s \in M - N\ s >_{\mathsf{mpo}} t$$

Theorem

$>_{\mathsf{mpo}}$ is reduction order if $>$ is well-founded

Definition

- weight function (w, w_0) consists of mapping $\mathsf{w}\colon \mathcal{F} \to \mathbb{N}$ and constant $w_0 > 0$ such that $\mathsf{w}(c) \geq w_0$ for all constants $c \in \mathcal{F}$
- weight of term t is

$$\mathsf{w}(t) = w_0 \cdot \Big(\sum_{x \in \mathsf{Var}(t)} |t|_x \Big) + \sum_{f \in \mathsf{FS}(t)} \mathsf{w}(f) \cdot |t|_f$$

- weight function (w, w_0) is admissible for precedence $>$ if

$$f > g \text{ for all } g \in \mathcal{F} \setminus \{f\}$$

whenever f is unary function symbol in $\mathcal{F}$ with $w(f) = 0$

$$\mathsf{w}(+) = \mathsf{w}(\mathsf{S}) = 0 \quad \mathsf{w}(0) = 1 \qquad \mathsf{S} > \circ > 0$$

Definition

- precedence is proper order $>$ on $\mathcal{F}$
- admissible weight function (w, w_0)
- relation $>_{\mathsf{kbo}}$ (Knuth-Bendix order) on terms:
  $s >_{\mathsf{kbo}} t$ if $|s|_x \geq |t|_x$ for all $x \in \mathcal{V}$ and either
  1. $w(s) > w(t)$,
  2. $w(s) = w(t)$ and either
     1. $\exists\, n > 0\ \exists\, x \in \mathcal{V}\ s = f^n(x)$ and $t = x$
     2. $s = f(s_1, \dots, s_n)$ and $t = f(t_1, \dots, t_n)$ and $\exists\, i$

        $$\forall\, j < i\ s_j = t_j \qquad s_i >_{\mathsf{kbo}} t_i$$
     3. $s = f(s_1, \dots, s_n)$ and $t = g(t_1, \dots, t_m)$ and $f > g$

Theorem

$>_{\mathsf{kbo}}$ is reduction order if $>$ is well-founded and (w, w_0) admissible

Theorem

- if $> \subseteq \sqsupset$ and (w, w_0) admissible then $>_{\mathsf{kbo}}\ \subseteq\ >_{\mathsf{kbo}\,\sqsupset}$ (incrementality)
- if $>$ is total then $>_{\mathsf{kbo}}$ is total on ground terms (well-order)
- following two problems are decidable:
  1. instance: terms s, t $\quad >\quad (\mathsf{w}, w_0)$
     question: $s >_{\mathsf{kbo}} t$?
  2. instance: terms s, t
     question: $\exists$ precedence $>$ and admissible (w, w_0) such that $s >_{\mathsf{kbo}} t$?

$$\begin{aligned} \mathsf{g}(\mathsf{g}(x)) &\to \mathsf{f}(x) \\ \mathsf{f}(\mathsf{g}(x)) &\to \mathsf{g}(\mathsf{f}(x)) \end{aligned} \qquad \mathsf{f} > \mathsf{g} \wedge \mathsf{w}(\mathsf{f}) = \mathsf{w}(\mathsf{g}) = 1$$

Definition

- well-founded monotone $\mathcal{F}$-algebra (WFMA) $(\mathcal{A}, >)$ is non-empty algebra $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f \in \mathcal{F}})$ together with well-founded order $>$ on A such that every $f_{\mathcal{A}}$ is strictly monotone in all coordinates:

$$f_{\mathcal{A}}(a_1, \dots, a_i, \dots, a_n) \ > \ f_{\mathcal{A}}(a_1, \dots, b, \dots, a_n)$$

  for all $a_1, \dots, a_n, b \in A$ and $i \in [1, n]$ with $a_i > b$
- binary relation $>_{\mathcal{A}}$ on terms:

$$s >_{\mathcal{A}} t \iff \underbrace{[\alpha]_{\mathcal{A}}(s)}_{} > [\alpha]_{\mathcal{A}}(t) \quad \text{for all assignments } \alpha$$

  interpretation of s in $\mathcal{A}$ under assignment α
- TRS $\mathcal{R}$ and WFMA $(\mathcal{A}, >)$ are compatible if $\mathcal{R}$ and $>_{\mathcal{A}}$ are compatible

Theorem

- $>_{\mathcal{A}}$ is reduction order for every WFMA $(\mathcal{A}, >)$
- TRS is terminating iff compatible with WFMA

Definition

TRS $\mathcal{R}$ is polynomially terminating if compatible with WFMA $(\mathcal{A}, >)$ such that

1. carrier of $\mathcal{A}$ is $\mathbb{N}$
2. $>$ is standard order on $\mathbb{N}$
3. $f_{\mathcal{A}}$ is polynomial for every f

$$
\begin{array}{rlrl}
x + 0 & \to x & 0 & := 1 \\
x + \mathsf{S}(y) & \to \mathsf{S}(x + y) & \mathsf{S}_{\mathcal{A}} & := \lambda x\,.\,x + 1 \\
x \times 0 & \to 0 & +_{\mathcal{A}} & := \lambda xy\,.\,x + 2y \\
x \times \mathsf{S}(y) & \to x \times y + x & \times_{\mathcal{A}} & := \lambda xy\,.\,(x+1)(y+1)^2
\end{array}
$$

History

- interpretation method — Turing 1949
- polynomial interpretations — Lankford 1975; Ben Cherifa, Lescanne 1987
- lexicographic path order — Schütte 1960; Dershowitz 1982; Kamin, Lévy 1980
- Knuth-Bendix order — Knuth, Bendix 1970; Dick, Kalmus, Martin 1990
- recursive decomposition order — Jouannaud, Lescanne, Reinig 1982

Remark

traditional termination methods yield simple termination