Advanced Topics in Term Rewriting
LVA 703610

http://cl-informatik.uibk.ac.at/teaching/ws06/attr/

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Derivational Complexity of TRSs

$(\mathcal{F}, \mathcal{R})$ a finitely branching and terminating TRS, $\mathcal{F}$ contains at least one constant

- ➡ let t be a ground term
- ➡ the derivation height function $\mathsf{dh}_{\mathcal{R}}$ is defined as

$$\mathsf{dh}_{\mathcal{R}}(s) = \max(\{n \mid \exists t \; s \to^n_{\mathcal{R}} t\})$$

the derivation height of s measures the maximal number of rewrite steps (aka complexity of $\mathcal{R}$) with initial term s

- ➡ the derivational complexity function dc_{RS} is defined as

$$\mathsf{dc}_{\mathcal{R}}(n) = \max(\{\mathsf{dh}_{\mathcal{R}}(s) \mid \mathsf{size}(s) \leqslant n\})$$

Well-known Results

finite signatures

Theorem ①

MPO induces primitive recursive derivational complexity

Hofbauer 1992

Theorem ②

LPO induces multiply recursive derivational complexity

Weiermann 1995

Theorem ③

KBO induces a 2-recursive upper bound, more precisely the derivational complexity function is bounded by $\mathsf{Ack}(\mathcal{O}(n), 0)$

Lepper 2001

all mentioned upper-bounds are tight

Application

Example

Find a TRS that is provable terminating via semantic labelling and KBO but not with semantic labelling on finite models and MPO.

How to show that a given TRS is not terminating via semantic labelling and MPO

Lemma

$\forall s, \mathsf{dh}_{\mathcal{R}}(s) = \mathsf{dh}_{\mathcal{R}_{\text{lab}} \cup \mathcal{D}ec(\succ)}(s)$ use first version

Idea

Use Theorem ① to conclude that MPO-termination induced primitive recursive derivational complexity, together with the above lemma

Polynomials induce double-exponential complexity

Theorem ④

polynomial interpretations induce double-exponential derivational complexity Hofbauer, Lautemann 1989

Lemma

$\forall$ $\mathcal{R}$ terminating via a polynomial interpretation
$\exists$ $c \in \mathbb{R}$, $c > 0$
$\forall$ terms s: $\mathsf{dh}_{\mathcal{R}}(s) \leq 2^{2^{c \cdot \mathsf{size}(s)}}$

Lemma

$\exists$ $\mathcal{R}$ terminating via a polynomial interpretation
$\exists$ $c \in \mathbb{R}$, $c > 0$
for infintely many terms s: $\mathsf{dh}_{\mathcal{R}}(s) \geq 2^{2^{c \cdot \mathsf{size}(s)}}$

Proof

consider

$$x + 0 \to x \qquad \mathsf{d}(0) \to 0 \qquad \mathsf{d}(\mathsf{S}(x)) \to \mathsf{S}(\mathsf{S}(\mathsf{d}(x)))$$
$$x + \mathsf{S}(y) \to \mathsf{S}(x + y) \qquad \mathsf{q}(0) \to 0 \qquad \mathsf{q}(\mathsf{S}(x)) \to \mathsf{q}(x) + \mathsf{S}(\mathsf{d}(x))$$

together with $\mathcal{A} = (\mathbb{N} - \{0,1\}, >)$ and $0_{\mathcal{A}} = 0$, $\mathsf{S}_{\mathcal{A}}(n) = n + 1$, $n +_{\mathcal{A}} m = n + 2m$, $\mathsf{d}_{\mathcal{A}}(n) = 3n$, $\mathsf{q}_{\mathcal{A}}(n) = n^3$

1. S defines the successor function
2. d defines the double function, i.e., $\mathsf{d}(\mathsf{S}^n(0)) \to^*_{\mathcal{R}} \mathsf{S}^{2n}(0)$
3. q defines the square function, i.e., $\mathsf{q}(\mathsf{S}^n(0)) \to^*_{\mathcal{R}} \mathsf{S}^{n^2}(0)$

To see item 3, we assume 1,2 and proceed by induction on n
Case $n = 0$: $\mathsf{q}(\mathsf{S}^0(0)) \to^*_{\mathcal{R}} \mathsf{S}^{0^2}(0)$

Case $n > 0$:

$$q(S^{n+1}(0)) \to_{\mathcal{R}} q(S^n(0)) + S(d(S^n(0))) \to^*_{\mathcal{R}} S^{n^2}(0) + S(d(S^n(0))) \to^*_{\mathcal{R}}$$
$$\to^*_{\mathcal{R}} S^{n^2}(0) + S(S^2\, n(0)) \to^*_{\mathcal{R}} S^{(n+1)^2}(0)$$

using items 1–3, we see

$$s_m := q^{m+1}(S^2(0)) \to^*_{\mathcal{R}} q(S^{2^{2^m}}(0)) \to^{2^{2^m}}_{\mathcal{R}} S^{2^{2^{m+1}}}(0)$$

hence

$$\mathrm{dh}_{\mathcal{R}}(s_m) \geqslant 2^{2^n} = 2^{2^{\mathrm{size}(s_m)-4}} \geqslant 2^{2^{c \cdot \mathrm{size}(s_m)}}$$

where $c \leqslant \frac{1}{5}$ and all $m \geqslant 1$. □

Conclusion

Proof Scheme

- find a mapping $\mathrm{I}\colon \mathcal{T}(\mathcal{F}) \to \mathbb{N}$ such that for all $s, t \in \mathcal{T}(\mathcal{F})$:

  $$s \to_{\mathcal{R}} t \text{ implies } \mathrm{I}(s) > \mathrm{I}(t)$$

  then $\mathrm{dh}_{\mathcal{R}}(s) \leqslant \mathrm{I}(s)$, and

  $$\mathrm{dc}_{\mathcal{R}}(n) \leq \max(\{\mathrm{I}(s) \mid \mathrm{size}(s) \leqslant n\})$$

Limitations

- consider $\mathcal{R}$ consisting of $a(b(x)) \to b(a(x))$ the system is polynomially terminating, with $\mathcal{A} = (\mathbb{N}, >)$ $a_{\mathcal{A}}(n) = 2n$, $b_{\mathcal{A}}(n) = n + 1$, $c_{\mathcal{A}} = 0$
  1. $2^n \cdot m = \mathrm{I}(a^n\, b^m\, c) \geqslant \mathrm{dh}_{\mathcal{R}}(a^n\, b^m\, c)$, but
  2. $\mathrm{dh}_{\mathcal{R}}(a^n\, b^m\, c) = n \cdot m$