Advanced Topics in Term Rewriting
LVA 703610

`http://cl-informatik.uibk.ac.at/teaching/ws06/attr/`

Georg Moser

office hours: Tuesday, 16:00–18:00 (3M09)

MPOs induce Primitive Recursive Derivational Complexities

Theorem Hofbauer 1992

if $\mathcal{R}$ is a finite TRS compatible with an instance of MPO, then $\mathsf{dc}_{\mathcal{R}}$ is primitively recursively bounded; moreover this bound is tight

Lemma lower-bound

for any primitive recursive function f, there exists a TRS $\mathcal{R}$, compatible with an instance of MPO, such that f is bounded by $\mathcal{R}$

Lemma upper-bound

if $\mathcal{R}$ is a finite TRS compatible with $>_{\mathsf{mpo}}$, then $\mathsf{dc}_{\mathcal{R}}$ is primitive recursively bounded

Analyse MPO

total precedence

let $\succ$ denote a total precedence, and assume $\mathcal{R}$ is compatible with $>_{\mathsf{mpo}}$

Definition

$$\begin{aligned}\mathsf{front}(s,f) \quad := \quad & \{u \in \mathcal{P}\mathsf{os}(s) \mid \mathsf{root}(s\mid_u) = g \succeq f \text{ and} \\ & \forall v < u \; f \succ h = \mathsf{root}(t\mid_v)\} \cup \\ & \cup \{u \in \mathcal{P}\mathsf{os}(s) \mid s\mid_u \in \mathcal{V}\}\end{aligned}$$

Lemma

1. $s >_{\mathsf{mpo}} t$ iff $\forall u \in \mathsf{front}(s, \mathsf{root}(s))\; s >_{\mathsf{mpo}} s\mid_u$
2. $f(s_1, \ldots, s_n) >_{\mathsf{mpo}} f(t_1, \ldots, t_n)$ iff $\{s_1, \ldots, s_n\}(>_{\mathsf{mpo}})_{\mathsf{mul}}\{t_1, \ldots, t_n\}$
3. for $\mathsf{root}(s) \succ f$ $f(s_1, \ldots, s_n) >_{\mathsf{mpo}} t$ iff $\exists i\; s_i \geq_{\mathsf{mpo}} t$

Preliminaries

for a preorder $\geqslant$, we define its lexicographic extension on lists of terms in nonincreasing order

$$\begin{aligned}&(s_1, \ldots, s_n)\; \mathsf{lex}(>)\; (t_1, \ldots, t_n) \iff n > 0 \text{ and} \\ &(s_1 > t_1 \text{ or } (s_1 = t_1 \text{ and } (s_2, \ldots, s_n)\; \mathsf{lex}(>)\; (t_2, \ldots, t_n)))\end{aligned}$$

Lemma

$\forall$ k, $\exists$ an order-preserving mapping $q_k \colon \mathbb{N}^k \to \mathbb{N}$ with

1. $q_k(n_1, \ldots, n_m) = q_k(n_{\pi(1)}, \ldots, n_{\pi(m)})$ for any permutation π
2. q_k is monotonic, and admits the subterm property
3. $q_k(n_1, \ldots, n_j, 0, \ldots, 0) > q_k(n_1, \ldots, n_j - 1, n_j - 1, \ldots, n_j - 1)$, if $n_1 \geqslant \cdots \geqslant n_j > 0$
4. $q_k(n_1, \ldots, n_m) > q_k(l_1, \ldots, l_m)$, if $(n_1, \ldots, n_m)\; \mathsf{lex}(>)\; (l_1, \ldots, l_m)$

Bounding the Evaluations

Lemma

let $\mathcal{A} = (\mathbb{N}, >)$ be WMA compatible with $\mathcal{R}$ and let p be a strictly monotonic unary function on $\mathbb{N}$ such that $p(n) \geqslant f_{\mathcal{A}}(n, \dots, n)$
Then we have $\mathsf{dc}_{\mathcal{R}}(n) \leqslant p^n(0)$

Proof

by induction on t, the base-case is trivial and for the step-case we obtain

$$\begin{aligned}
[\alpha]_{\mathcal{A}}(f(t_1, \dots, t_m)) &\leqslant f_{\mathcal{A}}([\alpha]_{\mathcal{A}}(t_1), \dots, [\alpha]_{\mathcal{A}}(t_m)) \\
&\leqslant f_{\mathcal{A}}(p^{\mathsf{size}(t_1)}(0), \dots, p^{\mathsf{size}(t_m)}(0)) \\
&\leqslant p(\max(\{p^{\mathsf{size}(t_1)}(0), \dots, p^{\mathsf{size}(t_m)}(0)\})) \\
&\leqslant p^{\mathsf{size}(t)}(0) \quad \square
\end{aligned}$$

A Well-founded Monotone Algebra

set

$$d := \max(\{\mathsf{depth}\ r \mid l \to r \in \mathcal{R}\}$$

let $\mathcal{A} = (\mathbb{N}, >)$ be an algebra equipped with the following interpretation:

$$\begin{aligned}
p_f(n) &:= \begin{cases} n + 1 & \text{if } f \text{ minimal} \\ \max(\{h_{\mathcal{A}}(n, \dots, n) \mid f > h\}) + n & \text{otherwise} \end{cases} \\
P_f(n) &:= \begin{cases} 1 & \text{if } n = 0 \\ 1 + p_f^d(P_f(n-1)) & \text{if } n > 0 \end{cases} \\
f_{\mathcal{A}}(n_1, \dots, n_m) &:= P_f(q_m(n_1, \dots, n_m))
\end{aligned}$$

Lemma

$\mathcal{A}$ is a WMA

Main Lemma

Lemma

let s be a term with $\text{depth}(s) < d$
assume that $\forall$ non-variable positions u in s, $f > \text{root}(s|_u)$
$\forall\, x \in \text{Var}(s)$: $m \geqslant [\alpha]_{\mathcal{A}}(x)$
Then $p_f^d(m) \geqslant [\alpha]_{\mathcal{A}}(t)$

Proof

induction on d □

Lemma

$s >_{\text{mpo}} t$ and $\text{depth}(t) \leqslant d$ we have $[\alpha]_{\mathcal{A}}(s) > [\alpha]_{\mathcal{A}}(t)$

Proof induction on $s = f(s_1, \ldots, s_n)$

Proof Ideas

1. only subterms at positions in *front*(f, s) are critical
2. exploit the montonicity and subterm property of $\mathcal{A}$
3. the definition of $\mathcal{A}$ induces that the evaluation remains unchanged by permutation of the arguments

Lemma upper-bound

if $\mathcal{R}$ is a finite TRS compatible with $>_{\text{mpo}}$, then $\text{dc}_{\mathcal{R}}$ is primitive recursively bounded

Proof

- ➡ $p(n) = \max\{f_{\mathcal{A}}(n, \ldots, n) \mid f \in \mathcal{F}\}$
- ➡ $\text{dc}_{\mathcal{R}}(n) \leqslant p^n(0)$
- ➡ the functions $f_{\mathcal{A}}$ are primitive recursive, hence p and its iterations is primitive recursive □