Advanced Topics in Term Rewriting
LVA 703610

http://cl-informatik.uibk.ac.at/teaching/ws06/attr/

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office hours: Tuesday, 16:00–18:00 (3M09)

Definition

- well-founded monotone $\mathcal{F}$-algebra (WFMA) $(\mathcal{A}, >)$ is non-empty algebra $\mathcal{A} = (A, \{f_\mathcal{A}\}_{f \in \mathcal{F}})$ together with well-founded order $>$ on A such that every $f_\mathcal{A}$ is strictly monotone in all coordinates:

$$f_\mathcal{A}(a_1, \ldots, a_i, \ldots, a_n) \ > \ f_\mathcal{A}(a_1, \ldots, b, \ldots, a_n)$$

  for all $a_1, \ldots, a_n, b \in A$ and $i \in [1, n]$ with $a_i > b$
- binary relation $>_\mathcal{A}$ on terms:

$$s >_\mathcal{A} t \iff \underbrace{[\alpha]_\mathcal{A}(s)}_{} > [\alpha]_\mathcal{A}(t) \quad \text{for all assignments } \alpha$$

  interpretation of s in $\mathcal{A}$ under assignment α

- TRS $\mathcal{R}$ and WFMA $(\mathcal{A}, >)$ are compatible if $\mathcal{R}$ and $>_\mathcal{A}$ are compatible

Theorem

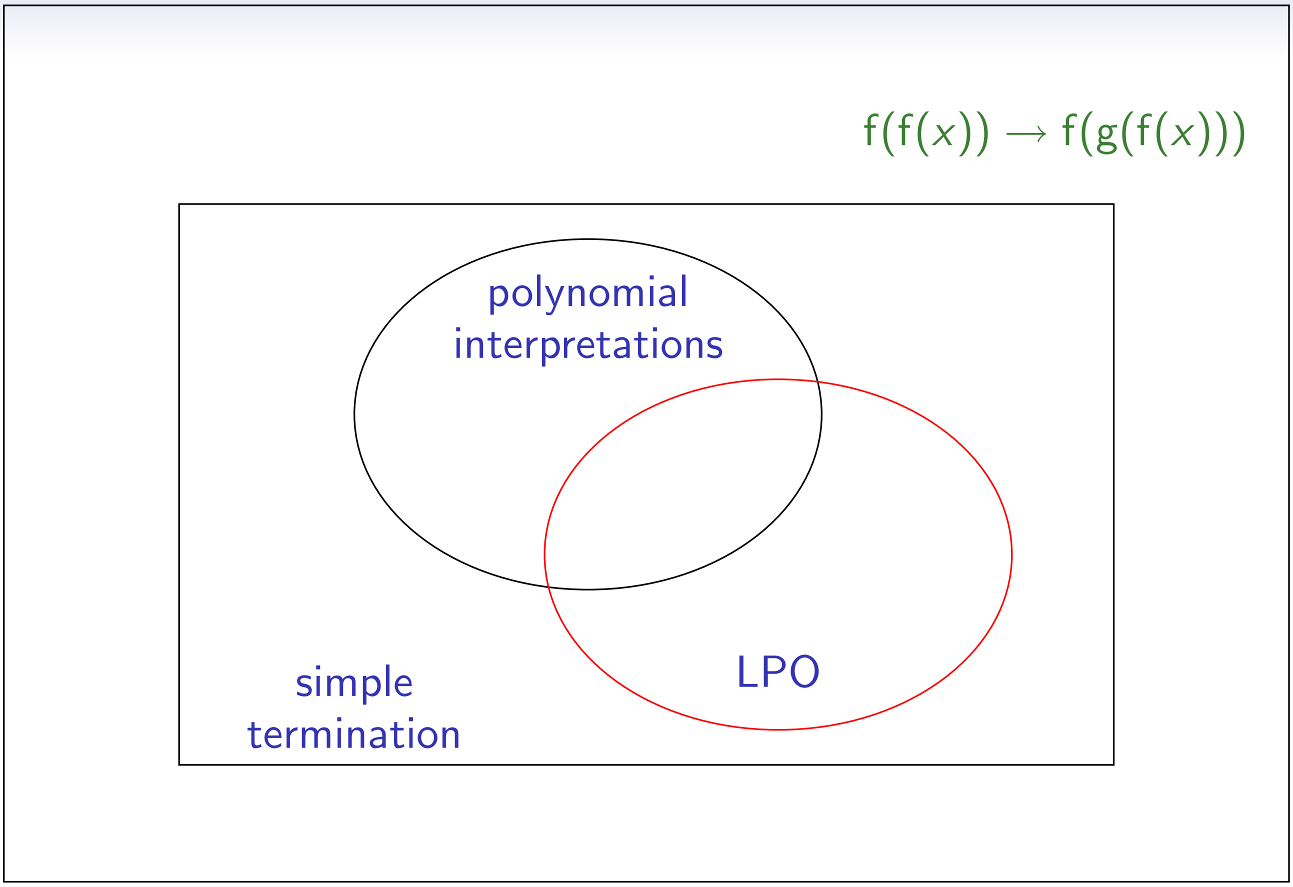
- $>_{\mathcal{A}}$ is reduction order for every WFMA $(\mathcal{A}, >)$
- TRS is terminating iff compatible with WFMA

Definition

TRS $\mathcal{R}$ is polynomially terminating if compatible with WFMA $(\mathcal{A}, >)$ such that

1. carrier of $\mathcal{A}$ is $\mathbb{N}$
2. $>$ is standard order on $\mathbb{N}$
3. $f_{\mathcal{A}}$ is polynomial for every f

$$
\begin{aligned}
x + 0 &\to x & 0 &:= 1 \\
x + \mathsf{S}(y) &\to \mathsf{S}(x + y) & \mathsf{S}_{\mathcal{A}} &:= \lambda x\,.\, x + 1 \\
x \times 0 &\to 0 & +_{\mathcal{A}} &:= \lambda xy\,.\, x + 2y \\
x \times \mathsf{S}(y) &\to x \times y + x & \times_{\mathcal{A}} &:= \lambda xy\,.\,(x+1)(y+1)^2
\end{aligned}
$$



Simple Termination

signatures are finite

Definition

- binary relation $>$ on terms is simplification order if
  1. closed under contexts
  2. closed under substitutions
  3. proper order
  4. subterm property $\quad u[t]_p > t$ when $p \neq \epsilon$
- TRS is simply terminating if compatible with simplification order
- TRS $\mathcal{E}\text{mb}$ consists of all rewrite rules

$$\mathsf{f}(x_1, \dots, x_n) \to x_i$$

- $\trianglerighteq \; = \; \to^*_{\mathcal{E}\text{mb}}$ $\qquad$ embedding

Theorem

simplification orders are well-founded

proof is based on Kruskal's Tree Theorem:

if $t_1, t_2, t_3, \dots$ is infinite sequence of ground terms then $t_i \trianglelefteq t_j$ for some $i < j$

Theorem

simply terminating TRSs are terminating

Lemma

TRS $\mathcal{R}$ is simply terminating iff $\mathcal{R} \cup \mathcal{E}\text{mb}$ is terminating

$$\mathsf{f}(\mathsf{f}(x)) \to \mathsf{f}(\mathsf{g}(\mathsf{f}(x))) \to_{\mathcal{E}\text{mb}} \mathsf{f}(\mathsf{f}(x))$$

TRS $\mathcal{R}$

$$\begin{aligned} \mathsf{fac}(0) &\to \mathsf{S}(0) & 0 + y &\to y \\ \mathsf{fac}(\mathsf{S}(x)) &\to \mathsf{S}(x) \times \mathsf{fac}(\mathsf{p}(\mathsf{S}(x))) & \mathsf{S}(x) + y &\to \mathsf{S}(x + y) \\ \mathsf{p}(\mathsf{S}(0)) &\to 0 & 0 \times y &\to 0 \\ \mathsf{p}(\mathsf{S}(\mathsf{S}(x))) &\to \mathsf{S}(\mathsf{p}(\mathsf{S}(x))) & \mathsf{S}(x) \times y &\to x \times y + y \end{aligned}$$

is not simply terminating:

$$\mathsf{fac}(\mathsf{S}(x)) \to \mathsf{S}(x) \times \mathsf{fac}(\mathsf{p}(\mathsf{S}(x))) \to_{\mathcal{E}\mathsf{mb}} \mathsf{fac}(\mathsf{p}(\mathsf{S}(x))) \to_{\mathcal{E}\mathsf{mb}} \mathsf{fac}(\mathsf{S}(x))$$

termination of $\mathcal{R}$ is intuitively clear:

value of argument of fac decreases in recursive call

Idea

idea: label fac by value of argument

Definition

signature $\mathcal{F}$ $\quad$ $\mathcal{F}$-algebra $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f \in \mathcal{F}})$

- labelling L for $\mathcal{F}$ $\quad$ $\forall f \in \mathcal{F}$ $\exists$sets of labels $L_f \subseteq A$
- labelled signature

  $$\mathcal{F}_{\text{lab}} = \{f_a \mid f \in \mathcal{F} \text{ and } a \in L_f\} \cup \{f \mid f \in \mathcal{F} \text{ and } L_f = \varnothing\}$$
- labelling ℓ for $\mathcal{A}$ is a labelling L for $\mathcal{F}$ and $\forall f \in \mathcal{F}(L_f \neq \varnothing \to \exists$ mappings $\ell_f \colon A^n \to L_f)$
- labelling function lab_α for every assignment α:

  $$\mathsf{lab}_\alpha(t) = \begin{cases} t & \text{if } t \in \mathcal{V} \\ f(\mathsf{lab}_\alpha(t_1), \dots, \mathsf{lab}_\alpha(t_n)) & \text{if } t = f(\overline{t}) \text{ and } L_f = \varnothing \\ f_a(\mathsf{lab}_\alpha(t_1), \dots, \mathsf{lab}_\alpha(t_n)) & \text{if } t = f(\overline{t}) \text{ and } L_f \neq \varnothing \end{cases}$$

  with $a = \ell_f([\alpha]_{\mathcal{A}}(t_1), \dots, [\alpha]_{\mathcal{A}}(t_n))$
- labelled TRS

  $$\mathcal{R}_{\text{lab}} = \{\mathsf{lab}_\alpha(l) \to \mathsf{lab}_\alpha(r) \mid l \to r \in \mathcal{R} \text{ and } \alpha \in A^{\mathcal{V}}\}$$

TRS $\mathcal{R}$

$$\begin{aligned} \mathsf{fac}_{0}(0) &\to \mathsf{S}(0) & 0 + y &\to y \\ \mathsf{fac}_{i+1}(\mathsf{S}(x)) &\to \mathsf{S}(x) \times \mathsf{fac}_{i}(\mathsf{p}(\mathsf{S}(x))) & \mathsf{S}(x) + y &\to \mathsf{S}(x + y) \\ \mathsf{p}(\mathsf{S}(0)) &\to 0 & 0 \times y &\to 0 \\ \mathsf{p}(\mathsf{S}(\mathsf{S}(x))) &\to \mathsf{S}(\mathsf{p}(\mathsf{S}(x))) & \mathsf{S}(x) \times y &\to x \times y + y \end{aligned}$$

➡ algebra $\mathcal{A}$ with carrier $\mathbb{N}$ and interpretations

$$\begin{aligned} 0_{\mathcal{A}} &= 0 & \mathsf{S}_{\mathcal{A}}(x) &= x + 1 & +_{\mathcal{A}}(x, y) &= x + y \\ \mathsf{fac}_{\mathcal{A}}(x) &= x! & \mathsf{p}_{\mathcal{A}}(x) &= \max\{x - 1, 0\} & \times_{\mathcal{A}}(x, y) &= xy \end{aligned}$$

➡ labelling LPO with $\mathsf{fac}_{i+1} > \mathsf{fac}_{i} > \times > + > \mathsf{p} > \mathsf{S}$

$$L_{\mathsf{fac}} = \mathbb{N} \text{ with } \ell_{\mathsf{fac}}(x) = x$$

$$L_{0} = L_{\mathsf{S}} = L_{\mathsf{p}} = L_{+} = L_{\times} = \varnothing$$

TRS $\mathcal{R}$

$$\mathsf{f}_2(\mathsf{f}_1(x)) \to \mathsf{f}_1(\mathsf{g}(\mathsf{f}_1(x)))$$

$$\mathsf{f}_2(\mathsf{f}_2(x)) \to \mathsf{f}_1(\mathsf{g}(\mathsf{f}_2(x)))$$

➡ algebra $\mathcal{A}$ with carrier $\{1, 2\}$ and interpretations

$$\mathsf{f}_{\mathcal{A}}(x) = 2 \qquad \mathsf{g}_{\mathcal{A}}(x) = 1$$

➡ labelling RPO with $\mathsf{f}_2 > \mathsf{f}_1 > \mathsf{g}$

$$L_{\mathsf{f}} = \{1, 2\} \text{ with} \ell_{\mathsf{f}}(x) = x \qquad L_{\mathsf{g}} = \varnothing$$

Theorem

∀ TRS $\mathcal{R}$
∀ non-empty model $\mathcal{A}$ of $\mathcal{R}$
∀ labelling ℓ for $\mathcal{A}$

$$\mathcal{R} \text{ is terminating} \iff \mathcal{R}_{\text{lab}} \text{ is terminating}$$