

# Advanced Topics in Term Rewriting

## LVA 703610

<http://cl-informatik.uibk.ac.at/teaching/ws06/attr/>

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office hours: **Tuesday, 16:00–18:00** (3M09)

### Theorem

- $>_{\mathcal{A}}$  is **reduction order** for every WFMA  $(\mathcal{A}, >)$
- TRS is terminating iff compatible with WFMA

### Definition

TRS  $\mathcal{R}$  is **polynomially terminating** if compatible with WFMA  $(\mathcal{A}, >)$  such that

- 1 carrier of  $\mathcal{A}$  is  $\mathbb{N}$
- 2  $>$  is standard order on  $\mathbb{N}$
- 3  $f_{\mathcal{A}}$  is polynomial for every  $f$

$$\begin{array}{ll}
 x + 0 \rightarrow x & 0 := 1 \\
 x + S(y) \rightarrow S(x + y) & S_{\mathcal{A}} := \lambda x. x + 1 \\
 x \times 0 \rightarrow 0 & +_{\mathcal{A}} := \lambda xy. x + 2y \\
 x \times S(y) \rightarrow x \times y + x & \times_{\mathcal{A}} := \lambda xy. (x + 1)(y + 1)^2
 \end{array}$$

### Definition

- **well-founded monotone  $\mathcal{F}$ -algebra (WFMA)**  $(\mathcal{A}, >)$  is non-empty algebra  $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f \in \mathcal{F}})$  together with well-founded order  $>$  on  $A$  such that every  $f_{\mathcal{A}}$  is **strictly monotone** in all coordinates:

$$f_{\mathcal{A}}(a_1, \dots, a_i, \dots, a_n) > f_{\mathcal{A}}(a_1, \dots, b, \dots, a_n)$$

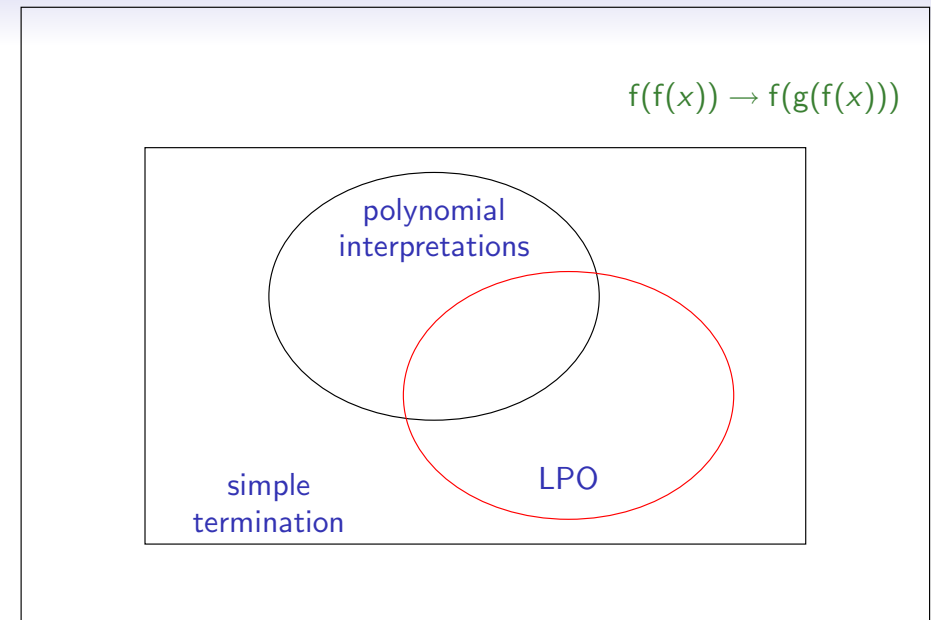
for all  $a_1, \dots, a_n, b \in A$  and  $i \in [1, n]$  with  $a_i > b$

- binary relation  $>_{\mathcal{A}}$  on terms:

$$s >_{\mathcal{A}} t \iff [\alpha]_{\mathcal{A}}(s) > [\alpha]_{\mathcal{A}}(t) \quad \text{for all assignments } \alpha$$

interpretation of  $s$  in  $\mathcal{A}$  under assignment  $\alpha$

- TRS  $\mathcal{R}$  and WFMA  $(\mathcal{A}, >)$  are **compatible** if  $\mathcal{R}$  and  $>_{\mathcal{A}}$  are compatible



## Simple Termination

signatures are finite

### Definition

- binary relation  $>$  on terms is **simplification order** if
  - 1 closed under contexts
  - 2 closed under substitutions
  - 3 proper order
  - 4 **subterm property**  $u[t]_p > t$  when  $p \neq \epsilon$
- TRS is **simply terminating** if compatible with simplification order
- TRS  $\mathcal{E}mb$  consists of all rewrite rules

$$f(x_1, \dots, x_n) \rightarrow x_i$$

- $\triangleright = \rightarrow_{\mathcal{E}mb}^*$  **embedding**

### TRS $\mathcal{R}$

$$\begin{array}{ll} \text{fac}(0) \rightarrow S(0) & 0 + y \rightarrow y \\ \text{fac}(S(x)) \rightarrow S(x) \times \text{fac}(p(S(x))) & S(x) + y \rightarrow S(x + y) \\ p(S(0)) \rightarrow 0 & 0 \times y \rightarrow 0 \\ p(S(S(x))) \rightarrow S(p(S(x))) & S(x) \times y \rightarrow x \times y + y \end{array}$$

is not simply terminating:

$$\text{fac}(S(x)) \rightarrow S(x) \times \text{fac}(p(S(x))) \rightarrow_{\mathcal{E}mb} \text{fac}(p(S(x))) \rightarrow_{\mathcal{E}mb} \text{fac}(S(x))$$

termination of  $\mathcal{R}$  is **intuitively** clear:

value of argument of **fac** decreases in recursive call

### Idea

**idea:** label **fac** by value of argument

### Theorem

simplification orders are well-founded

proof is based on **Kruskal's Tree Theorem**:

*if  $t_1, t_2, t_3, \dots$  is infinite sequence of ground terms then  $t_i \not\leq t_j$  for some  $i < j$*

### Theorem

simply terminating TRSs are terminating

### Lemma

TRS  $\mathcal{R}$  is simply terminating iff  $\mathcal{R} \cup \mathcal{E}mb$  is terminating

$$f(f(x)) \rightarrow f(g(f(x))) \rightarrow_{\mathcal{E}mb} f(f(x))$$

### Definition

signature  $\mathcal{F}$   $\mathcal{F}$ -algebra  $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f \in \mathcal{F}})$

- labelling  $L$  for  $\mathcal{F}$   $\forall f \in \mathcal{F} \exists$  sets of labels  $L_f \subseteq A$
- **labelled signature**

$$\mathcal{F}_{lab} = \{f_a \mid f \in \mathcal{F} \text{ and } a \in L_f\} \cup \{f \mid f \in \mathcal{F} \text{ and } L_f = \emptyset\}$$
- labelling  $\ell$  for  $\mathcal{A}$  is a labelling  $L$  for  $\mathcal{F}$  and  $\forall f \in \mathcal{F} (L_f \neq \emptyset \rightarrow \exists$  mappings  $\ell_f: A^n \rightarrow L_f)$
- labelling function **lab $_{\alpha}$**  for every assignment  $\alpha$ :
 
$$\text{lab}_{\alpha}(t) = \begin{cases} t & \text{if } t \in \mathcal{V} \\ f(\text{lab}_{\alpha}(t_1), \dots, \text{lab}_{\alpha}(t_n)) & \text{if } t = f(\bar{t}) \text{ and } L_f = \emptyset \\ f_a(\text{lab}_{\alpha}(t_1), \dots, \text{lab}_{\alpha}(t_n)) & \text{if } t = f(\bar{t}) \text{ and } L_f \neq \emptyset \end{cases}$$
 with  $a = \ell_f([\alpha]_{\mathcal{A}}(t_1), \dots, [\alpha]_{\mathcal{A}}(t_n))$
- **labelled TRS**

$$\mathcal{R}_{lab} = \{\text{lab}_{\alpha}(l) \rightarrow \text{lab}_{\alpha}(r) \mid l \rightarrow r \in \mathcal{R} \text{ and } \alpha \in A^{\mathcal{V}}\}$$

TRS  $\mathcal{R}$ 

$$\begin{array}{ll} \text{fac}_0(0) \rightarrow S(0) & 0 + y \rightarrow y \\ \text{fac}_{i+1}(S(x)) \rightarrow S(x) \times \text{fac}_i(p(S(x))) & S(x) + y \rightarrow S(x + y) \\ p(S(0)) \rightarrow 0 & 0 \times y \rightarrow 0 \\ p(S(S(x))) \rightarrow S(p(S(x))) & S(x) \times y \rightarrow x \times y + y \end{array}$$

→ algebra  $\mathcal{A}$  with carrier  $\mathbb{N}$  and interpretations

$$\begin{array}{lll} 0_{\mathcal{A}} = 0 & S_{\mathcal{A}}(x) = x + 1 & +_{\mathcal{A}}(x, y) = x + y \\ \text{fac}_{\mathcal{A}}(x) = x! & p_{\mathcal{A}}(x) = \max\{x - 1, 0\} & \times_{\mathcal{A}}(x, y) = xy \end{array}$$

→ labelling **LPO** with  $\text{fac}_{i+1} > \text{fac}_i > \times > + > p > S$

$$\begin{array}{l} L_{\text{fac}} = \mathbb{N} \text{ with } \ell_{\text{fac}}(x) = x \\ L_0 = L_S = L_p = L_+ = L_{\times} = \emptyset \end{array}$$

TRS  $\mathcal{R}$ 

$$\begin{array}{l} f_2(f_1(x)) \rightarrow f_1(g(f_1(x))) \\ f_2(f_2(x)) \rightarrow f_1(g(f_2(x))) \end{array}$$

→ algebra  $\mathcal{A}$  with carrier  $\{1, 2\}$  and interpretations

$$f_{\mathcal{A}}(x) = 2 \quad g_{\mathcal{A}}(x) = 1$$

→ labelling **RPO** with  $f_2 > f_1 > g$

$$L_f = \{1, 2\} \text{ with } \ell_f(x) = x \quad L_g = \emptyset$$

**Theorem**

∀ TRS  $\mathcal{R}$

∀ **non-empty model**  $\mathcal{A}$  of  $\mathcal{R}$

∀ labelling  $\ell$  for  $\mathcal{A}$

$$\mathcal{R} \text{ is terminating} \iff \mathcal{R}_{\text{lab}} \text{ is terminating}$$