

Advanced Topics in Term Rewriting

LVA 703610

<http://cl-informatik.uibk.ac.at/teaching/ws06/attr/>

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TRS \mathcal{R}

$$f_2(f_1(x)) \rightarrow f_1(g(f_1(x)))$$

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➔ algebra \mathcal{A} with carrier $\{1, 2\}$ and interpretations

$$f_{\mathcal{A}}(x) = 2$$

$$g_{\mathcal{A}}(x) = 1$$

➔ labelling **MPO** with $f_2 > f_1 > g$

$$L_f = \{1, 2\} \text{ with } \ell_f(x) = x$$

$$L_g = \emptyset$$

Observation

➔ MPO is compatible with \mathcal{R}

➔ does this establish **(simple) termination**?

- infinite sequence n_1, n_2, n_3, \dots of natural numbers is **increasing** if $\exists i < j$ such that $n_i \leq n_j$

Lemma

every infinite sequence of natural numbers contains increasing subsequence

Dickson's Lemma

\forall infinite sequence e_1, e_2, e_3, \dots of n -tuples of natural numbers
 $\exists i < j$ such that

- $e_i = (a_1, \dots, a_n)$
- $e_j = (b_1, \dots, b_n)$
- $a_k \leq b_k \quad \forall 1 \leq k \leq n$

Proof

induction on n using the above lemma

let \succ be a proper order, an infinite sequence a_1, a_2, a_3, \dots of elements of A is

- **good** if $\exists i < j$ such that $a_i \preceq a_j$
- **bad** if not good
- **chain** if $a_i \preceq a_{i+1} \quad \forall i \geq 1$
- **antichain** if $a_i \not\preceq a_j$ and $a_j \not\preceq a_i \quad \forall j > i \geq 1$

Definition

a **WPO** is a proper order \succ on A such that every infinite sequence of elements of A is good

Theorem

following statements are **equivalent**:

- \succ is a well-partial order (WPO)
- every proper order extending \succ (including \succ itself) is well-founded
- every infinite sequence contains chain

Theorem

$\triangleright_{\text{emb}}$ is a WPO on ground terms over finite signatures

Proof

we construct a **minimal bad sequence** \mathbf{t} :

given t_1, \dots, t_{n-1}

define t_1, \dots, t_n

such that t_n minimal for all bad sequence with initial segment t_1, \dots, t_n

- 1 $\forall i$, let $\{t_1^i, \dots, t_m^i\}$ denote the set of immediate subterms of t_i , let S denote the set of all these subterms
- 2 $\triangleright_{\text{emb}}$ is a WPO on S and any infinite sequence of elements of S contains a chain
- 3 let \mathbf{t}^* denote a subsequence of \mathbf{t} such that $t_i^* = f(t_1^i, \dots, t_m^i)$

- 4 the sequence \mathbf{e} with components $e_i = (t_1^i, \dots, t_m^i)$ contains a chain

$$e_{\varphi(1)} \triangleleft_{\text{emb}} e_{\varphi(2)} \triangleleft_{\text{emb}} \dots \triangleleft_{\text{emb}} e_{\varphi(i)} \triangleleft_{\text{emb}} e_{\varphi(i+1)} \dots$$

- 5 we obtain

$$t_{\varphi(1)}^* \triangleleft_{\text{emb}} t_{\varphi(2)}^* \triangleleft_{\text{emb}} \dots \triangleleft_{\text{emb}} t_{\varphi(i)}^* \triangleleft_{\text{emb}} t_{\varphi(i+1)}^* \dots$$

□

Kruskal's Theorem - general version

\succ_{emb} is a WPO on set of ground terms if \succ is a WPO on signature

Definitionproper order \succ on signature

→ TRS $\mathcal{E}mb(\succ)$ is extension of $\mathcal{E}mb$ with all rules

$$f(x_1, \dots, x_n) \rightarrow g(x_{i_1}, \dots, x_{i_m})$$

with $f \succ g$ and $1 \leq i_1 < \dots < i_m \leq n$ whenever $m \geq 1$

→ $\succ_{emb} = \rightarrow_{\mathcal{E}mb(\succ)}^*$ (homeomorphic embedding)

→ $\succ_{emb} = \rightarrow_{\mathcal{E}mb(\succ)}^+$

Lemma

homeomorphic embedding generalises embedding, as

$$\rightarrow_{\mathcal{E}mb(\emptyset)}^* = \rightarrow_{\mathcal{E}mb}^*$$

proof of Kruskal's Tree Theorem depends on **Higman's Lemma**

proper order \succ on set A

Definition

relation \succ^* on **words** over A :

$w_1 \succ^* w_2$ if $a_1 \cdots a_n = w_1 \neq w_2 = b_1 \cdots b_m$ and either

1 $m = 0$

2 $n \geq m > 0$ and $\exists 1 \leq i_1 < \dots < i_m \leq n \quad \forall 1 \leq j \leq m$
 $a_{i_j} \succ b_j$

Lemma

\succ^* is proper order on A^*

Higman's Lemma

\succ^* is a WPO on A^* if \succ is WPO on A

Definition

simplification order is **rewrite order** that contains \succ_{emb} for some WPO \succ

Theorem

simplification orders are well-founded

Lemma

- $>_{\text{lpo}}$ ($>_{\text{kbo}}$) are simplification orders if $>$ is **WPO**
- $>_{\text{lpo}}$ ($>_{\text{kbo}}$) are reduction orders if $>$ is **well-founded**

Theorem

TRS is **simply terminating** if compatible with simplification order

- simply terminating TRSs are terminating