Advanced Topics in Term Rewriting
LVA 703610

http://cl-informatik.uibk.ac.at/teaching/ws06/attr/

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office hours: Tuesday, 16:00–18:00 (3M09)

TRS $\mathcal{R}$

$$f_2(f_1(x)) \to f_1(g(f_1(x)))$$
$$f_2(f_2(x)) \to f_1(g(f_2(x)))$$

- algebra $\mathcal{A}$ with carrier $\{1,2\}$ and interpretations

$$f_{\mathcal{A}}(x) = 2 \qquad g_{\mathcal{A}}(x) = 1$$

- labelling MPO with $f_2 > f_1 > g$

$$L_f = \{1,2\} \text{ with } \ell_f(x) = x \qquad L_g = \varnothing$$

Observation

- MPO is compatible with $\mathcal{R}$
- does this estabhlish (simple) termination?

Simple Termination revisited

signatures are finite

Definition

simplification order is rewrite order with subterm property

Lemma

- $\rhd_{\mathrm{emb}}$ $(:= \to^{+}_{\mathcal{E}\mathrm{mb}})$ is smallest simplification order
- rewrite order is simplification order iff it contains $\rhd_{\mathrm{emb}}$

Theorem

simplification orders are well-founded
proof is based on finite version of Kruskal's Tree Theorem

Definition

infinite sequence $t_1, t_2, t_3, \ldots$ of terms is self-embedding if $\exists\, i < j$ such that $t_i \trianglelefteq_{\mathrm{emb}} t_j$

Kruskal's Theorem - finite version

every infinite sequence of ground terms is self-embedding

signature $\mathcal{F}$ f g a

$$f(f(a)) \quad f(g(f(a))) \quad f(f(a)) \quad f(g(f(a))) \quad \cdots$$

TRS $\mathcal{R}$

$$f(f(x)) \to f(g(f(x))) \qquad f(x) \to x$$
$$g(x) \to x$$

➡ infinite sequence $n_1, n_2, n_3, \ldots$ of natural numbers is increasing if $\exists\ i < j$ such that $n_i \leq n_j$

Lemma

every infinite sequence of natural numbers contains increasing subsequence

Dickson's Lemma

$\forall$ infinite sequence $e_1, e_2, e_3, \ldots$ of n-tuples of natural numbers $\exists\ i < j$ such that

- ➡ $e_i = (a_1, \ldots, a_n)$
- ➡ $e_j = (b_1, \ldots, b_n)$
- ➡ $a_k \leq b_k \quad \forall\ 1 \leq k \leq n$

Proof

induction on n using the above lemma

let $\succ$ be a proper order, an infinite sequence $a_1, a_2, a_3, \ldots$ of elements of A is

- ➡ good if $\exists\ i < j$ such that $a_i \preccurlyeq a_j$
- ➡ bad if not good
- ➡ chain if $a_i \preccurlyeq a_{i+1} \quad \forall\ i \geq 1$
- ➡ antichain if $a_i \not\preccurlyeq a_j$ and $a_j \not\preccurlyeq a_i \quad \forall\ j > i \geq 1$

Definition

a WPO is a proper order $\succ$ on A such that every infinite sequence of elements of A is good

Theorem

following statements are equivalent:

- ➡ $\succ$ is a well-partial order (WPO)
- ➡ every proper order extending $\succ$ (including $\succ$ itself) is well-founded
- ➡ every infinite sequence contains chain

Theorem

$\rhd_{\text{emb}}$ is a WPO on ground terms over finite signatures

Proof

we construct a minimal bad sequence $\mathbf{t}$:

given $t_1, \ldots, t_{n-1}$
define $t_1, \ldots, t_n$

such that t_n minimal for all bad sequence with initial segment $t_1, \ldots, t_n$

1. $\forall i$, let $\{t^i_1, \ldots, t^i_m\}$ denote the set of immediate subterms of t_i, let S denote the set of all these subterms
2. $\rhd_{\text{emb}}$ is a WPO on S and any infinite sequence of elements of S contains a chain
3. let $\mathbf{t}^*$ denote a subsequence of $\mathbf{t}$ such that $t^*_i = f(t^i_1, \ldots, t^i_m)$

4. the sequence $\mathbf{e}$ with components $e_i = (t^i_1, \ldots, t^i_m)$ contains a chain

$$e_{\varphi(1)} \lhd_{\text{emb}} e_{\varphi(2)} \lhd_{\text{emb}} \cdots \lhd_{\text{emb}} e_{\varphi(i)} \lhd_{\text{emb}} e_{\varphi(i+1)} \cdots$$

5. we obtain

$$t^*_{\varphi(1)} \lhd_{\text{emb}} t^*_{\varphi(2)} \lhd_{\text{emb}} \cdots \lhd_{\text{emb}} t^*_{\varphi(i)} \lhd_{\text{emb}} t^*_{\varphi(i+1)} \cdots$$

□

Kruskal's Theorem - general version

$\succ_{\text{emb}}$ is a WPO on set of ground terms if $\succ$ is a WPO on signature

Definition proper order $\succ$ on signature

- TRS $\mathcal{E}\mathsf{mb}(\succ)$ is extension of $\mathcal{E}\mathsf{mb}$ with all rules

$$\mathsf{f}(x_1, \dots, x_n) \to \mathsf{g}(x_{i_1}, \dots, x_{i_m})$$

  with $\mathsf{f} \succ \mathsf{g}$ and $1 \leqslant i_1 < \cdots < i_m \leqslant n$ whenever $m \geqslant 1$

- $\succcurlyeq_{\mathsf{emb}} \; = \; \to^*_{\mathcal{E}\mathsf{mb}(\succ)}$ (homeomorphic embedding)
- $\succ_{\mathsf{emb}} \; = \; \to^+_{\mathcal{E}\mathsf{mb}(\succ)}$

Lemma

homemorphic embedding generalises embedding, as

$\to^*_{\mathcal{E}\mathsf{mb}(\varnothing)} = \to^*_{\mathcal{E}\mathsf{mb}}$

proof of Kruskal's Tree Theorem depends on Higman's Lemma

proper order $\succ$ on set A

Definition

relation $\succ^*$ on words over A:

$w_1 \succ^* w_2$ if $a_1 \cdots a_n = w_1 \neq w_2 = b_1 \cdots b_m$ and either

1. $m = 0$
2. $n \geqslant m > 0$ and $\exists\, 1 \leqslant i_1 < \cdots < i_m \leqslant n \quad \forall\, 1 \leqslant j \leqslant m$

   $a_{i_j} \succcurlyeq b_j$

Lemma

$\succ^*$ is proper order on A^*

Higman's Lemma

$\succ^*$ is a WPO on A^* if $\succ$ is WPO on A

Definition

simplification order is rewrite order that contains $\succ_{\mathsf{emb}}$ for some WPO $\succ$

Theorem

simplification orders are well-founded

Lemma

- $>_{\mathsf{lpo}}$ ($>_{\mathsf{kbo}}$) are simplification orders if $>$ is WPO
- $>_{\mathsf{lpo}}$ ($>_{\mathsf{kbo}}$) are reduction orders if $>$ is well-founded

Theorem

TRS is simply terminating if compatible with simplification order

- simply terminating TRSs are terminating