Advanced Topics in Term Rewriting
LVA 703610

http://cl-informatik.uibk.ac.at/teaching/ws06/attr/

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Semantic Labelling

Theorem

$\forall$ TRS $\mathcal{R}$
$\forall$ non-empty model $\mathcal{A}$ of $\mathcal{R}$
$\forall$ labelling ℓ for $\mathcal{A}$

$$\mathcal{R} \text{ is terminating} \iff \mathcal{R}_{\mathrm{lab}} \text{ is terminating}$$

Observation

this version of semantic labelling is useless to prove termination of

$$\mathsf{f}(\mathsf{a},\mathsf{b},x) \to \mathsf{f}(x,x,x) \qquad \mathsf{a} \to \mathsf{c}$$
$$\mathsf{f}(x,y,z) \to \mathsf{c} \qquad \mathsf{b} \to \mathsf{c}$$

Definition

- weakly monotone $\mathcal{F}$-algebra WMA $(\mathcal{A}, \succ)$ is non-empty algebra $\mathcal{A} = (A, \{f_\mathcal{A}\}_{f \in \mathcal{F}})$ together with proper order $\succ$ on A such that every $f_\mathcal{A}$ is weakly monotone in all coordinates:

$$f_\mathcal{A}(a_1, \ldots, a_i, \ldots, a_n) \succeq f_\mathcal{A}(a_1, \ldots, b, \ldots, a_n)$$

  for all $a_1, \ldots, a_n, b \in A$ and $i \in [1, n]$ with $a_i \succ b$
- binary relation $\succeq_\mathcal{A}$ on terms:

$$s \succeq_\mathcal{A} t \iff \underbrace{[\alpha]_\mathcal{A}(s)}_{\text{interpretation of } s \text{ in } \mathcal{A} \text{ under assignment } \alpha} \succeq [\alpha]_\mathcal{A}(t) \quad \text{for all assignments } \alpha$$

- WMA $(\mathcal{A}, \succ)$ is quasi-model of TRS $\mathcal{R}$ if $\succeq_\mathcal{A}$ and $\mathcal{R}$ are compatible
- a WMA $(\mathcal{A}, \succ)$ is well-founded if $\succ$ is well-founded

signature $\mathcal{F}$ WMA $(\mathcal{A}, \succ)$ with $\mathcal{A} = (A, \{f_\mathcal{A}\}_{f \in \mathcal{F}})$

Definition

- labelling L for $\mathcal{F}$ $\forall f \in \mathcal{F}$ $\exists$sets of labels $L_f \subseteq A$
- labelled signature

$$\mathcal{F}_{\text{lab}} = \{f_a \mid f \in \mathcal{F} \text{ and } a \in L_f\} \cup \{f \mid f \in \mathcal{F} \text{ and } L_f = \varnothing\}$$

- labelling ℓ for $\mathcal{A}$ is a labelling L for $\mathcal{F}$ and $\forall f \in \mathcal{F}$ $(L_f \neq \varnothing \rightarrow \exists$ mappings $\ell_f \colon A^n \rightarrow L_f)$
- ℓ is weakly monotone if all labeling functions are weakly monotone (in all arguments)
- TRS $\mathcal{D}ec(\succ)$ consists of all rules

$$f_a(x_1, \ldots, x_n) \rightarrow f_b(x_1, \ldots, x_n)$$

  with $a, b \in L_f$ such that $a \succ b$

TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{a},\mathsf{b},x) \to \mathsf{f}(x,x,x) \qquad \mathsf{a} \to \mathsf{c}$$
$$\mathsf{f}(x,y,z) \to \mathsf{c} \qquad \mathsf{b} \to \mathsf{c}$$

➡ quasi-model $\mathcal{A}$

carrier: $\{0,1,2\}$ order: $1 > 0$, $2 > 0$

interpretation: $\mathsf{f}_{\mathcal{A}}(x,y,z) = \mathsf{c}_{\mathcal{A}} = 0$ $\quad \mathsf{a}_{\mathcal{A}} = 1$ $\quad \mathsf{b}_{\mathcal{A}} = 2$

➡ labelling

$L_{\mathsf{f}} = \{0,1\} \qquad L_{\mathsf{a}} = L_{\mathsf{b}} = L_{\mathsf{c}} = \varnothing$

$$\ell_{\mathsf{f}}(x,y,z) = \begin{cases} 1 & \text{if } x = 1 \text{ and } y = 2 \\ 0 & \text{otherwise} \end{cases}$$

TRS $\mathcal{R}_{\mathrm{lab}} \cup \mathcal{D}ec(\succ)$ MPO with $\mathsf{f}_1 > \mathsf{f}_0 > \mathsf{c}$ $\quad \mathsf{a} > \mathsf{c}$ $\quad \mathsf{b} > \mathsf{c}$

$$\mathsf{f}_1(\mathsf{a},\mathsf{b},x) \to \mathsf{f}_0(x,x,x) \qquad \mathsf{a} \to \mathsf{c}$$
$$\mathsf{f}_1(x,y,z) \to \mathsf{c} \qquad \mathsf{b} \to \mathsf{c}$$
$$\mathsf{f}_0(x,y,z) \to \mathsf{c} \qquad \mathsf{f}_1(x,y,z) \to \mathsf{f}_0(x,y,z)$$

Theorem

$\forall$ TRS $\mathcal{R}$
$\forall$ well-founded quasi-model $(\mathcal{A},\succ)$ of $\mathcal{R}$
$\forall$ weakly monotone labelling ℓ for $(\mathcal{A},\succ)$

$$\mathcal{R} \text{ is terminating} \iff \mathcal{R}_{\mathrm{lab}} \cup \mathcal{D}ec(\succ) \text{ is terminating}$$

Theorem

$\forall$ terminating TRS $\mathcal{R}$
$\exists$ well-founded quasi-model $(\mathcal{A},\succ)$ of $\mathcal{R}$
$\exists$ weakly monotone labelling ℓ for $(\mathcal{A},\succ)$

$$\mathcal{R}_{\mathrm{lab}} \cup \mathcal{D}ec(\succ) \text{ is precedence terminating}$$