Advanced Topics in Term Rewriting
LVA 703610

http://cl-informatik.uibk.ac.at/teaching/ws06/attr/

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Semantic Labelling

Theorem

$\forall$ TRS $\mathcal{R}$
$\forall$ non-empty model $\mathcal{A}$ of $\mathcal{R}$
$\forall$ labelling ℓ for $\mathcal{A}$

$$\mathcal{R} \text{ is terminating} \iff \mathcal{R}_{\text{lab}} \text{ is terminating}$$

Observation

this version of semantic labelling is useless to prove termination of

$$\mathsf{f}(\mathsf{a},\mathsf{b},x) \to \mathsf{f}(x,x,x) \qquad \mathsf{a} \to \mathsf{c}$$
$$\mathsf{f}(x,y,z) \to \mathsf{c} \qquad \mathsf{b} \to \mathsf{c}$$

Definition

- weakly monotone $\mathcal{F}$-algebra WMA $(\mathcal{A}, \succ)$ is non-empty algebra $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f\in\mathcal{F}})$ together with proper order $\succ$ on A such that every $f_{\mathcal{A}}$ is weakly monotone in all coordinates:

  $$f_{\mathcal{A}}(a_1, \dots, a_i, \dots, a_n) \succeq f_{\mathcal{A}}(a_1, \dots, b, \dots, a_n)$$

  for all $a_1, \dots, a_n, b \in A$ and $i \in [1, n]$ with $a_i \succ b$
- binary relation $\succeq_{\mathcal{A}}$ on terms:

  $$s \succeq_{\mathcal{A}} t \iff \underbrace{[\alpha]_{\mathcal{A}}(s)} \succeq [\alpha]_{\mathcal{A}}(t) \quad \text{for all assignments } \alpha$$

  interpretation of s in $\mathcal{A}$ under assignment α
- WMA $(\mathcal{A}, \succ)$ is quasi-model of TRS $\mathcal{R}$ if $\succeq_{\mathcal{A}}$ and $\mathcal{R}$ are compatible
- a WMA $(\mathcal{A}, \succ)$ is well-founded if $\succ$ is well-founded

signature $\mathcal{F}$ WMA $(\mathcal{A}, \succ)$ with $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f\in\mathcal{F}})$

Definition

- labelling L for $\mathcal{F}$ $\forall f \in \mathcal{F}\ \exists$sets of labels $L_f \subseteq A$
- labelled signature

  $$\mathcal{F}_{\text{lab}} = \{f_a \mid f \in \mathcal{F} \text{ and } a \in L_f\} \cup \{f \mid f \in \mathcal{F} \text{ and } L_f = \varnothing\}$$
- labelling ℓ for $\mathcal{A}$ is a labelling L for $\mathcal{F}$ and $\forall f \in \mathcal{F}\ (L_f \neq \varnothing \to \exists \text{ mappings } \ell_f \colon A^n \to L_f)$
- ℓ is weakly monotone if all labeling functions are weakly monotone (in all arguments)
- TRS $\mathcal{D}ec(\succ)$ consists of all rules

  $$f_a(x_1, \dots, x_n) \to f_b(x_1, \dots, x_n)$$

  with $a, b \in L_f$ such that $a \succ b$

TRS $\mathcal{R}$

$$\mathsf{f}(\mathsf{a},\mathsf{b},x) \to \mathsf{f}(x,x,x) \qquad \mathsf{a} \to \mathsf{c}$$
$$\mathsf{f}(x,y,z) \to \mathsf{c} \qquad \mathsf{b} \to \mathsf{c}$$

- ➡ quasi-model $\mathcal{A}$

  carrier: $\{0,1,2\}$ order: $1 > 0,\ 2 > 0$

  interpretation: $\mathsf{f}_{\mathcal{A}}(x,y,z) = \mathsf{c}_{\mathcal{A}} = 0 \quad \mathsf{a}_{\mathcal{A}} = 1 \quad \mathsf{b}_{\mathcal{A}} = 2$

- ➡ labelling

  $L_{\mathsf{f}} = \{0,1\} \quad L_{\mathsf{a}} = L_{\mathsf{b}} = L_{\mathsf{c}} = \varnothing$

  $$\ell_{\mathsf{f}}(x,y,z) = \begin{cases} 1 & \text{if } x = 1 \text{ and } y = 2 \\ 0 & \text{otherwise} \end{cases}$$

TRS $\mathcal{R}_{\text{lab}} \cup \mathcal{D}ec(\succ)$ MPO with $\mathsf{f}_1 > \mathsf{f}_0 > \mathsf{c} \quad \mathsf{a} > \mathsf{c} \quad \mathsf{b} > \mathsf{c}$

$$\mathsf{f}_1(\mathsf{a},\mathsf{b},x) \to \mathsf{f}_0(x,x,x) \qquad \mathsf{a} \to \mathsf{c}$$
$$\mathsf{f}_1(x,y,z) \to \mathsf{c} \qquad \mathsf{b} \to \mathsf{c}$$
$$\mathsf{f}_0(x,y,z) \to \mathsf{c} \qquad \mathsf{f}_1(x,y,z) \to \mathsf{f}_0(x,y,z)$$

Theorem

$\forall$ TRS $\mathcal{R}$
$\forall$ well-founded quasi-model $(\mathcal{A},\succ)$ of $\mathcal{R}$
$\forall$ weakly monotone labelling ℓ for $(\mathcal{A},\succ)$

$$\mathcal{R} \text{ is terminating} \iff \mathcal{R}_{\text{lab}} \cup \mathcal{D}ec(\succ) \text{ is terminating}$$

Theorem

$\forall$ terminating TRS $\mathcal{R}$
$\exists$ well-founded quasi-model $(\mathcal{A},\succ)$ of $\mathcal{R}$
$\exists$ weakly monotone labelling ℓ for $(\mathcal{A},\succ)$

$$\mathcal{R}_{\text{lab}} \cup \mathcal{D}ec(\succ) \text{ is precedence terminating}$$