Advanced Topics in Term Rewriting
LVA 703610

http://cl-informatik.uibk.ac.at/teaching/ws06/attr/

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office hours: Tuesday, 16:00–18:00 (3M09)

Dependency Pairs

Observation

$\forall$ non-terminating TRS $\exists$ $\underbrace{\text{minimal}}$ non-terminating term

all proper subterms are terminating

➡ $\mathcal{T}_\infty$ is set of all minimal non-terminating terms

Lemma

$\forall t \in \mathcal{T}_\infty\ \exists (l \to r) \in \mathcal{R}\ \exists \sigma\ \exists$non-variable subterm u of r

$$t \xrightarrow{>\epsilon}{}^{*}_{\mathcal{R}} l\sigma \xrightarrow{\epsilon}_{\mathcal{R}} r\sigma \trianglerighteq u\sigma \in \mathcal{T}_\infty$$

Corollary

every term in $\mathcal{T}_\infty$ has defined root symbol

Lemma

$\forall t \in \mathcal{T}_\infty\ \exists (l \to r) \in \mathcal{R}\ \exists \sigma\ \exists$non-variable subterm u of r with $u \ntrianglelefteq l$

$$t \xrightarrow{>\epsilon}{}^*_{\mathcal{R}} l\sigma \xrightarrow{\epsilon}_{\mathcal{R}} r\sigma \trianglerighteq u\sigma \in \mathcal{T}_\infty$$

Definition

tentative

$\mathcal{S} = \{l \to u \mid l \to r \in \mathcal{R},\ u \trianglelefteq r$ with defined $\mathrm{root}(u),\ u \ntrianglelefteq l\}$

Lemma

$\forall t \in \mathcal{T}_\infty\ \exists (l \to u) \in \mathcal{S}\ \exists \sigma \qquad t \xrightarrow{>\epsilon}{}^*_{\mathcal{R}} l\sigma \xrightarrow{\epsilon}_{\mathcal{S}} u\sigma \in \mathcal{T}_\infty$

get rid of position constraints by marking root symbols of terms in rewrite rules of $\mathcal{S}$

TRS $\mathcal{R}$ over signature $\mathcal{F}$

- ➡ $\mathcal{F}^\sharp = \mathcal{F} \cup \{f^\sharp \mid f$ is defined symbol of $\mathcal{R}\}$
- ➡ if $t = f(t_1, \dots, t_n)$ with f defined then $t^\sharp = f^\sharp(t_1, \dots, t_n)$
- ➡ $\mathcal{T}^\sharp_\infty = \{t^\sharp \mid t \in \mathcal{T}_\infty\}$

$\mathrm{DP}(\mathcal{R}) = \{\underbrace{l^\sharp \to u^\sharp}_{\text{dependency pair}} \mid (l \to r) \in \mathcal{R},\ u \trianglelefteq r$ with defined $\mathrm{root}(u),\ u \ntrianglelefteq l\}$

Lemma

$\forall\ s \in \mathcal{T}_\infty\ \exists\ t, u \in \mathcal{T}_\infty \qquad s^\sharp \to^*_{\mathcal{R}} t^\sharp \to_{\mathrm{DP}(\mathcal{R})} u^\sharp$

Corollary

$\forall\ s \in \mathcal{T}^\sharp_\infty\ \exists\ t, u \in \mathcal{T}^\sharp_\infty \qquad s \to^*_{\mathcal{R}} t \to_{\mathrm{DP}(\mathcal{R})} u$

rewrite rules

$$
\begin{aligned}
0 - y &\to 0 \\
x - 0 &\to x \\
S(x) - S(y) &\to x - y \\
0 \div S(y) &\to 0 \\
S(x) \div S(y) &\to S((x - y) \div S(y))
\end{aligned}
$$

dependency pairs

$$
\begin{aligned}
S(x) -^\sharp S(y) &\to x -^\sharp y \\
S(x) \div^\sharp S(y) &\to (x - y) \div^\sharp S(y) \\
S(x) \div^\sharp S(y) &\to x -^\sharp y
\end{aligned}
$$

Theorem

$\forall$ non-terminating TRS $\mathcal{R}$ $\exists$ infinite rewrite sequence

$$t_1 \to_{\mathcal{R}}^* t_2 \to_{\mathsf{DP}(\mathcal{R})} t_3 \to_{\mathcal{R}}^* t_4 \to_{\mathsf{DP}(\mathcal{R})} t_5 \cdots$$

with $t_i \in \mathcal{T}_\infty^\sharp$ for all i

Definition

reduction pair $(\succsim, \succ)$ consists of preorder $\succsim$ and well-founded order $\succ$ such that

1. $\succsim$ is closed under contexts and substitutions
2. $\succ$ is closed under substitutions
3. $\succsim \cdot \succ \cdot \succsim \subseteq \succ$

Theorem

TRS $\mathcal{R}$ is terminating if $\exists$ reduction pair $(\succsim, \succ)$ such that

1. $l \succsim r \quad \forall\ (l \to r) \in \mathcal{R}$
2. $l \succ r \quad \forall\ (l \to r) \in \mathsf{DP}(\mathcal{R})$

- weakly monotone $\mathcal{F}$-algebra WMA $(\mathcal{A}, \succ)$ is non-empty algebra $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f \in \mathcal{F}})$ together with proper order $\succ$ on A such that every $f_{\mathcal{A}}$ is weakly monotone in all coordinates
- binary relations $\succ_{\mathcal{A}}$, $\succeq_{\mathcal{A}}$ on terms:

$$s \succ_{\mathcal{A}} t \iff [\alpha]_{\mathcal{A}}(s) \succ [\alpha]_{\mathcal{A}}(t)$$
$$s \succeq_{\mathcal{A}} t \iff [\alpha]_{\mathcal{A}}(s) \succeq [\alpha]_{\mathcal{A}}(t)$$
for all assignments α

Lemma

$\forall$ well-founded WMA $(\mathcal{A}, \succ)$ $(\succeq_{\mathcal{A}}, \succ_{\mathcal{A}})$ is reduction pair

$\mathcal{A}$ domain $\mathbb{N}$ order $>$

$$0_{\mathcal{A}} = 0 \qquad \mathsf{S}_{\mathcal{A}}(n) = n + 1$$
$$-_{\mathcal{A}}(n, m) = \div_{\mathcal{A}}(n, m) = -^{\sharp}_{\mathcal{A}}(n, m) = \div^{\sharp}_{\mathcal{A}}(n, m) = n$$

constraints $\mathcal{R}$

$$\begin{aligned} 0 - y &\succeq_{\mathcal{A}} 0 \\ x - 0 &\succeq_{\mathcal{A}} x \\ \mathsf{S}(x) - \mathsf{S}(y) &\succeq_{\mathcal{A}} x - y \\ 0 \div \mathsf{S}(y) &\succeq_{\mathcal{A}} 0 \\ \mathsf{S}(x) \div \mathsf{S}(y) &\succeq_{\mathcal{A}} \mathsf{S}((x - y) \div \mathsf{S}(y)) \end{aligned}$$

constraints $\mathsf{DP}(\mathcal{R})$

$$\begin{aligned} \mathsf{S}(x) -^{\sharp} \mathsf{S}(y) &\succ_{\mathcal{A}} x -^{\sharp} y \\ \mathsf{S}(x) \div^{\sharp} \mathsf{S}(y) &\succ_{\mathcal{A}} (x - y) \div^{\sharp} \mathsf{S}(y) \\ \mathsf{S}(x) \div^{\sharp} \mathsf{S}(y) &\succ_{\mathcal{A}} x -^{\sharp} y \end{aligned}$$

$\mathcal{A}$ domain $\mathbb{N}$ order $>$

$$0_{\mathcal{A}} = 0 \qquad \mathsf{S}_{\mathcal{A}}(n) = n + 1$$
$$-_{\mathcal{A}}(n, m) = \div_{\mathcal{A}}(n, m) = -^{\sharp}_{\mathcal{A}}(n, m) = \div^{\sharp}_{\mathcal{A}}(n, m) = n$$

- reduction pair $(\succeq_{\mathcal{A}}, \succ_{\mathcal{A}})$