Advanced Topics in Term Rewriting
LVA 703610

`http://cl-informatik.uibk.ac.at/teaching/ws06/attr/`

Georg Moser

office hours: Tuesday, 16:00–18:00 (3M09)

Dependency Pairs

Observation

$\forall$ non-terminating TRS $\exists$ $\underbrace{\text{minimal}}$ non-terminating term

all proper subterms are terminating

- ➡ $\mathcal{T}_\infty$ is set of all minimal non-terminating terms

Lemma

$\forall t \in \mathcal{T}_\infty \ \exists (l \to r) \in \mathcal{R} \ \exists \sigma \ \exists$non-variable subterm u of r

$$t \xrightarrow{>\epsilon}{}^{*}_{\mathcal{R}} l\sigma \xrightarrow{\epsilon}_{\mathcal{R}} r\sigma \trianglerighteq u\sigma \in \mathcal{T}_\infty$$

Corollary

every term in $\mathcal{T}_\infty$ has defined root symbol

Lemma

$\forall t \in \mathcal{T}_\infty \ \exists (l \to r) \in \mathcal{R} \ \exists \sigma \ \exists$non-variable subterm u of r with $u \ntrianglelefteq l$

$$t \xrightarrow{>\epsilon}{}^{*}_{\mathcal{R}} l\sigma \xrightarrow{\epsilon}_{\mathcal{R}} r\sigma \trianglerighteq u\sigma \in \mathcal{T}_\infty$$

Definition

tentative

$\mathcal{S} = \{l \to u \mid l \to r \in \mathcal{R},\ u \trianglelefteq r$ with defined root(u), $u \ntrianglelefteq l\}$

Lemma

$\forall t \in \mathcal{T}_\infty \ \exists (l \to u) \in \mathcal{S} \ \exists \sigma \qquad t \xrightarrow{>\epsilon}{}^{*}_{\mathcal{R}} l\sigma \xrightarrow{\epsilon}_{\mathcal{S}} u\sigma \in \mathcal{T}_\infty$

get rid of position constraints by marking root symbols of terms in rewrite rules of $\mathcal{S}$

TRS $\mathcal{R}$ over signature $\mathcal{F}$

- ➡ $\mathcal{F}^\sharp = \mathcal{F} \cup \{f^\sharp \mid f$ is defined symbol of $\mathcal{R}\}$
- ➡ if $t = f(t_1, \dots, t_n)$ with f defined then $t^\sharp = f^\sharp(t_1, \dots, t_n)$
- ➡ $\mathcal{T}^\sharp_\infty = \{t^\sharp \mid t \in \mathcal{T}_\infty\}$

$\mathsf{DP}(\mathcal{R}) = \{\underbrace{l^\sharp \to u^\sharp} \mid (l \to r) \in \mathcal{R},\ u \trianglelefteq r$ with defined root(u), $u \ntrianglelefteq l\}$

dependency pair

Lemma

$\forall\ s \in \mathcal{T}_\infty \ \exists\ t, u \in \mathcal{T}_\infty \qquad s^\sharp \to^{*}_{\mathcal{R}} t^\sharp \to_{\mathsf{DP}(\mathcal{R})} u^\sharp$

Corollary

$\forall\ s \in \mathcal{T}^\sharp_\infty \ \exists\ t, u \in \mathcal{T}^\sharp_\infty \qquad s \to^{*}_{\mathcal{R}} t \to_{\mathsf{DP}(\mathcal{R})} u$

rewrite rules

$$
\begin{aligned}
0 - y &\to 0 \\
x - 0 &\to x \\
\mathsf{S}(x) - \mathsf{S}(y) &\to x - y \\
0 \div \mathsf{S}(y) &\to 0 \\
\mathsf{S}(x) \div \mathsf{S}(y) &\to \mathsf{S}((x - y) \div \mathsf{S}(y))
\end{aligned}
$$

dependency pairs

$$
\begin{aligned}
\mathsf{S}(x) -^\sharp \mathsf{S}(y) &\to x -^\sharp y \\
\mathsf{S}(x) \div^\sharp \mathsf{S}(y) &\to (x - y) \div^\sharp \mathsf{S}(y) \\
\mathsf{S}(x) \div^\sharp \mathsf{S}(y) &\to x -^\sharp y
\end{aligned}
$$

Theorem

$\forall$ non-terminating TRS $\mathcal{R}$ $\exists$ infinite rewrite sequence

$$t_1 \to^*_{\mathcal{R}} t_2 \to_{\mathsf{DP}(\mathcal{R})} t_3 \to^*_{\mathcal{R}} t_4 \to_{\mathsf{DP}(\mathcal{R})} t_5 \cdots$$

with $t_i \in \mathcal{T}^\sharp_\infty$ for all i

Definition

reduction pair $(\succsim, \succ)$ consists of preorder $\succsim$ and well-founded order $\succ$ such that

1. $\succsim$ is closed under contexts and substitutions
2. $\succ$ is closed under substitutions
3. $\succsim \cdot \succ \cdot \succsim \subseteq \succ$

Theorem

TRS $\mathcal{R}$ is terminating if $\exists$ reduction pair $(\succsim, \succ)$ such that

1. $l \succsim r \quad \forall\, (l \to r) \in \mathcal{R}$
2. $l \succ r \quad \forall\, (l \to r) \in \mathsf{DP}(\mathcal{R})$

- weakly monotone $\mathcal{F}$-algebra WMA $(\mathcal{A}, \succ)$ is non-empty algebra $\mathcal{A} = (A, \{f_\mathcal{A}\}_{f \in \mathcal{F}})$ together with proper order $\succ$ on A such that every $f_\mathcal{A}$ is weakly monotone in all coordinates
- binary relations $\succ_\mathcal{A}$, $\succeq_\mathcal{A}$ on terms:

$$
\begin{aligned}
s \succ_\mathcal{A} t &\iff [\alpha]_\mathcal{A}(s) \succ [\alpha]_\mathcal{A}(t) \\
s \succeq_\mathcal{A} t &\iff [\alpha]_\mathcal{A}(s) \succeq [\alpha]_\mathcal{A}(t)
\end{aligned}
\quad \text{for all assignments } \alpha
$$

Lemma

$\forall$ well-founded WMA $(\mathcal{A}, \succ)$ $(\succeq_\mathcal{A}, \succ_\mathcal{A})$ is reduction pair

$\mathcal{A}$ domain $\mathbb{N}$ order $>$

$0_\mathcal{A} = 0 \qquad \mathsf{S}_\mathcal{A}(n) = n + 1$

$$-_\mathcal{A}(n,m) = \div_\mathcal{A}(n,m) = -^\sharp_\mathcal{A}(n,m) = \div^\sharp_\mathcal{A}(n,m) = n$$

constraints $\mathcal{R}$

$$
\begin{aligned}
0 - y &\succeq_\mathcal{A} 0 \\
x - 0 &\succeq_\mathcal{A} x \\
\mathsf{S}(x) - \mathsf{S}(y) &\succeq_\mathcal{A} x - y \\
0 \div \mathsf{S}(y) &\succeq_\mathcal{A} 0 \\
\mathsf{S}(x) \div \mathsf{S}(y) &\succeq_\mathcal{A} \mathsf{S}((x - y) \div \mathsf{S}(y))
\end{aligned}
$$

constraints $\mathsf{DP}(\mathcal{R})$

$$
\begin{aligned}
\mathsf{S}(x) -^\sharp \mathsf{S}(y) &\succ_\mathcal{A} x -^\sharp y \\
\mathsf{S}(x) \div^\sharp \mathsf{S}(y) &\succ_\mathcal{A} (x - y) \div^\sharp \mathsf{S}(y) \\
\mathsf{S}(x) \div^\sharp \mathsf{S}(y) &\succ_\mathcal{A} x -^\sharp y
\end{aligned}
$$

$\mathcal{A}$ domain $\mathbb{N}$ order $>$

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$$-_\mathcal{A}(n,m) = \div_\mathcal{A}(n,m) = -^\sharp_\mathcal{A}(n,m) = \div^\sharp_\mathcal{A}(n,m) = n$$

- reduction pair $(\succeq_\mathcal{A}, \succ_\mathcal{A})$