Advanced Topics in Term Rewriting
LVA 703610

http://cl-informatik.uibk.ac.at/teaching/ws06/attr/

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office hours: Tuesday, 16:00–18:00 (3M09)

Dependency Pairs - Example

rewrite rules

$$\begin{aligned} 0 - y &\to 0 \\ x - 0 &\to x \\ \mathsf{S}(x) - \mathsf{S}(y) &\to x - y \\ 0 \div \mathsf{S}(y) &\to 0 \\ \mathsf{S}(x) \div \mathsf{S}(y) &\to \mathsf{S}((x - y) \div \mathsf{S}(y)) \end{aligned}$$

dependency pairs

$$\begin{aligned} \mathsf{S}(x) -^\sharp \mathsf{S}(y) &\to x -^\sharp y \\ \mathsf{S}(x) \div^\sharp \mathsf{S}(y) &\to (x - y) \div^\sharp \mathsf{S}(y) \\ \mathsf{S}(x) \div^\sharp \mathsf{S}(y) &\to x -^\sharp y \end{aligned}$$

LPO ?

simplify constraints by using argument filtering

Definition

➡ argument filtering is mapping π such that for every n-ary function symbol $f \in \mathcal{F}^\sharp$ one of following alternatives holds:

1. $\pi(f) = i$ with $i \in \{1, \dots, n\}$
2. $\pi(f) = [i_1, \dots, i_m]$ with $1 \leq i_1 \leq \cdots \leq i_m \leq n$

➡ $$\pi(t) = \begin{cases} t & \text{if } t \text{ is variable} \\ \pi(t_i) & \text{if } t = f(t_1, \dots, t_n) \text{ and } \boxed{1} \\ f(\pi(t_{i_1}), \dots, \pi(t_{i_m})) & \text{if } t = f(t_1, \dots, t_n) \text{ and } \boxed{2} \end{cases}$$

Theorem

TRS $\mathcal{R}$ is terminating if $\exists$ argument filtering π, $\exists$ reduction pair $(\gtrsim, \succ)$ such that

1. $\pi(l) \gtrsim \pi(r) \quad \forall\ l \to r \in \mathcal{R}$
2. $\pi(l) \succ \pi(r) \quad \forall\ l \to r \in \mathrm{DP}(\mathcal{R})$

Example - revisited

rewrite rules

$$\begin{aligned} 0 - y &\to 0 \\ x - 0 &\to x \\ \mathsf{S}(x) - \mathsf{S}(y) &\to x - y \\ 0 \div \mathsf{S}(y) &\to 0 \\ \mathsf{S}(x) \div \mathsf{S}(y) &\to \mathsf{S}((x - y) \div \mathsf{S}(y)) \end{aligned}$$

dependency pairs

$$\begin{aligned} \mathsf{S}(x) -^\sharp \mathsf{S}(y) &\to x -^\sharp y \\ \mathsf{S}(x) \div^\sharp \mathsf{S}(y) &\to (x - y) \div^\sharp \mathsf{S}(y) \\ \mathsf{S}(x) \div^\sharp \mathsf{S}(y) &\to x -^\sharp y \end{aligned}$$

argument filtering $\qquad \pi(-) = 1$

LPO $\qquad \div > \mathsf{S} \qquad \div^\sharp > -^\sharp$

Simplified Constraints

constraints $\mathcal{R}$

$$\begin{aligned} 0 &>_{\mathsf{lpo}} 0 \\ x &>_{\mathsf{lpo}} x \\ \mathsf{S}(x) &>_{\mathsf{lpo}} x \\ 0 \div \mathsf{S}(y) &>_{\mathsf{lpo}} 0 \\ \mathsf{S}(x) \div \mathsf{S}(y) &>_{\mathsf{lpo}} \mathsf{S}(x) \div \mathsf{S}(y) \end{aligned}$$

constraints $\mathsf{DP}(\mathcal{R})$

$$\begin{aligned} \mathsf{S}(x) -^\sharp \mathsf{S}(y) &>_{\mathsf{lpo}} x -^\sharp y \\ \mathsf{S}(x) \div^\sharp \mathsf{S}(y) &>_{\mathsf{lpo}} x \div^\sharp \mathsf{S}(y) \\ \mathsf{S}(x) \div^\sharp \mathsf{S}(y) &>_{\mathsf{lpo}} x -^\sharp y \end{aligned}$$

reduction pair $(\geq_{\mathsf{lpo}}, >_{\mathsf{lpo}})$

Example - extended

rewrite rules

$$\begin{aligned} 0 - y &\to 0 \\ x - 0 &\to x \\ \mathsf{S}(x) - \mathsf{S}(y) &\to x - y \\ 0 \div \mathsf{S}(y) &\to 0 \\ \mathsf{S}(x) \div \mathsf{S}(y) &\to \mathsf{S}((x - y) \div \mathsf{S}(y)) \\ 0 + y &\to y \\ \mathsf{S}(x) + y &\to \mathsf{S}(x + y) \\ (x - y) - z &\to x - (y + z) \end{aligned}$$

dependency pairs

$$\begin{aligned} \mathsf{S}(x) -^\sharp \mathsf{S}(y) &\to x -^\sharp y \\ \mathsf{S}(x) \div^\sharp \mathsf{S}(y) &\to (x - y) \div^\sharp \mathsf{S}(y) \\ \mathsf{S}(x) \div^\sharp \mathsf{S}(y) &\to x -^\sharp y \\ \mathsf{S}(x) +^\sharp y &\to x +^\sharp y \\ (x - y) -^\sharp z &\to x -^\sharp (y + z) \\ (x - y) -^\sharp z &\to y +^\sharp z \end{aligned}$$

Dependency Pairs - revisited

Corollary

$\forall$ non-terminating TRS $\mathcal{R}$ $\exists$ infinite rewrite sequence

$$\mathcal{T}^{\sharp}_{\infty} \ni t_1 \to^*_{\mathcal{R}} t_2 \to_{\mathsf{DP}(\mathcal{R})} t_3 \to^*_{\mathcal{R}} t_4 \to_{\mathsf{DP}(\mathcal{R})} \cdots$$

Assumption

all TRSs are finite

Lemma

$\forall$ non-terminating TRS $\mathcal{R}$ $\exists$ infinite rewrite sequence

$$\mathcal{T}^{\sharp}_{\infty} \ni t_1 \to^*_{\mathcal{R}} t_2 \to_{\mathcal{C}} t_3 \to^*_{\mathcal{R}} t_4 \to_{\mathcal{C}} \cdots \qquad \mathcal{C}\text{-minimal}$$

where all dependency pairs in $\mathcal{C} \subseteq \mathsf{DP}(\mathcal{R})$ are used infinitely often

Definition

dependency graph $\mathsf{DG}(\mathcal{R})$

nodes: $l \to r$ for every $(l \to r) \in \mathsf{DP}(\mathcal{R})$

arrows: $l_1 \to r_1 \longrightarrow l_2 \to r_2$

if $\exists$ substitutions σ, τ such that $r_1\sigma \to^*_{\mathcal{R}} l_2\tau$

Theorem

TRS $\mathcal{R}$ is terminating if

$$\forall \text{ cycle } \mathcal{C} \in \mathsf{DG}(\mathcal{R}) \qquad \neg \exists\, \mathcal{C} - \text{minimal rewrite sequence}$$

Remark

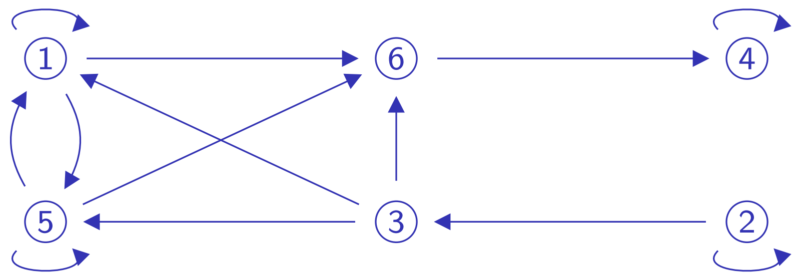
dependency graph is not computable in general, but good approximations exist rewrite rules

$$\begin{aligned} 0 - y &\to 0 & \mathsf{S}(x) \div \mathsf{S}(y) &\to \mathsf{S}((x - y) \div \mathsf{S}(y)) \\ x - 0 &\to x & 0 + y &\to y \\ \mathsf{S}(x) - \mathsf{S}(y) &\to x - y & \mathsf{S}(x) + y &\to \mathsf{S}(x + y) \\ 0 \div \mathsf{S}(y) &\to 0 & (x - y) - z &\to x - (y + z) \end{aligned}$$

dependency pairs

① $\mathsf{S}(x) -^\sharp \mathsf{S}(y) \to x -^\sharp y$
② $\mathsf{S}(x) \div^\sharp \mathsf{S}(y) \to (x - y) \div^\sharp \mathsf{S}(y)$
③ $\mathsf{S}(x) \div^\sharp \mathsf{S}(y) \to x -^\sharp y$
④ $\mathsf{S}(x) +^\sharp y \to x +^\sharp y$
⑤ $(x - y) -^\sharp z \to x -^\sharp (y + z)$
⑥ $(x - y) -^\sharp z \to y +^\sharp z$

dependency graph



5 cycles {①}, {③}, {④}, {⑤}, {①, ⑤}