Advanced Topics in Term Rewriting
LVA 703610

`http://cl-informatik.uibk.ac.at/teaching/ws06/attr/`

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office hours: Tuesday, 16:00–18:00 (3M09)

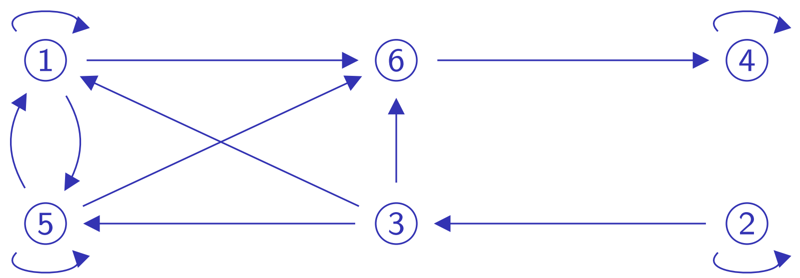
rewrite rules

$$
\begin{array}{rcl@{\qquad}rcl}
0 - y &\to& 0 & \mathsf{S}(x) \div \mathsf{S}(y) &\to& \mathsf{S}((x-y) \div \mathsf{S}(y)) \\
x - 0 &\to& x & 0 + y &\to& y \\
\mathsf{S}(x) - \mathsf{S}(y) &\to& x - y & \mathsf{S}(x) + y &\to& \mathsf{S}(x+y) \\
0 \div \mathsf{S}(y) &\to& 0 & (x-y) - z &\to& x - (y+z)
\end{array}
$$

dependency pairs

$$
\begin{array}{lrcl@{\qquad}lrcl}
① & \mathsf{S}(x) -^\sharp \mathsf{S}(y) &\to& x -^\sharp y & ④ & \mathsf{S}(x) +^\sharp y &\to& x +^\sharp y \\
② & \mathsf{S}(x) \div^\sharp \mathsf{S}(y) &\to& (x-y) \div^\sharp \mathsf{S}(y) & ⑤ & (x-y) -^\sharp z &\to& x -^\sharp (y+z) \\
③ & \mathsf{S}(x) \div^\sharp \mathsf{S}(y) &\to& x -^\sharp y & ⑥ & (x-y) -^\sharp z &\to& y +^\sharp z
\end{array}
$$

dependency graph



5 cycles {①}, {②}, {④}, {⑤}, {①, ⑤}

Theorem

$\forall$ non-terminating TRS $\mathcal{R}$ $\exists$ cycle $\mathcal{C}$ in DG($\mathcal{R}$)
$\exists$ $\mathcal{C}$-minimal rewrite sequence $t_1 \rightarrow^*_{\mathcal{R}} t_2 \rightarrow_{\mathcal{C}} t_3 \rightarrow^*_{\mathcal{R}} t_4 \rightarrow_{\mathcal{C}} \cdots$

Idea

project each dependency symbol in $\mathcal{C}$ to fixed argument position

$$\pi(t_1) \rightarrow^*_{\mathcal{R}} \pi(t_2) \;\; ? \;\; \pi(t_3) \rightarrow^*_{\mathcal{R}} \pi(t_4) \;\; ? \;\; \cdots$$

Observation

- $\pi(t_1)$ is terminating with respect to $\rightarrow_{\mathcal{R}}$ (because $t_1 \in \mathcal{T}^{\sharp}_{\infty}$)
- and also with respect to $\rightarrow_{\mathcal{R}} \cup \rhd$ (recall $\rhd \cdot \rightarrow_{\mathcal{R}} \subseteq \rightarrow_{\mathcal{R}} \cdot \rhd$)

Idea

require: $\forall\, l \rightarrow r \in \mathcal{C}$ $\pi(l) \trianglerighteq \pi(r)$ and $\exists\, l \rightarrow r \in \mathcal{C}$ $\pi(l) \rhd \pi(r)$

Subterm Criterion

Definition

- simple projection for cycle $\mathcal{C}$ in DG($\mathcal{R}$) is mapping π that assigns to every n-ary dependency pair symbol $f^{\sharp}$ in $\mathcal{C}$ one of its argument positions
- extension of π to terms in $\mathcal{T}^{\sharp}$: $\pi(f^{\sharp}(t_1, \ldots, t_n)) = t_{\pi(f^{\sharp})}$

Theorem

if $\exists$ simple projection π for cycle $\mathcal{C}$ in DG($\mathcal{R}$) such that

1. $\forall\, l \rightarrow r \in \mathcal{C}$ $\pi(l) = \pi(r)$ or $\pi(l) \rhd \pi(r)$ $\quad \pi(\mathcal{C}) \subseteq \trianglerighteq$
2. $\exists\, l \rightarrow r \in \mathcal{C}$ $\pi(l) \rhd \pi(r)$ $\quad \pi(\mathcal{C}) \cap \rhd \neq \varnothing$

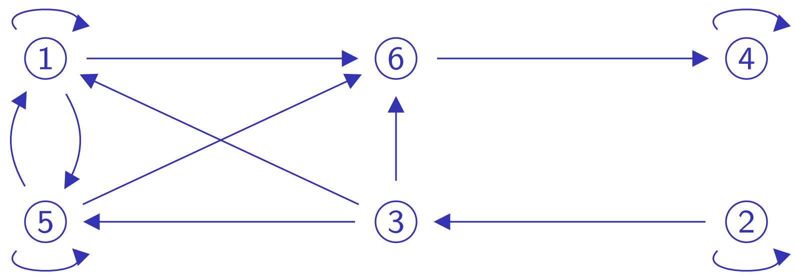
then $\neg\, \exists$ $\mathcal{C}$-minimal rewrite sequence rewrite rules

$$
\begin{array}{rcl@{\qquad}rcl}
0 - y & \to & 0 & \mathsf{S}(x) \div \mathsf{S}(y) & \to & \mathsf{S}((x - y) \div \mathsf{S}(y)) \\
x - 0 & \to & x & 0 + y & \to & y \\
\mathsf{S}(x) - \mathsf{S}(y) & \to & x - y & \mathsf{S}(x) + y & \to & \mathsf{S}(x + y) \\
0 \div \mathsf{S}(y) & \to & 0 & (x - y) - z & \to & x - (y + z)
\end{array}
$$

dependency pairs

① $\mathsf{S}(x) -^\sharp \mathsf{S}(y) \to x -^\sharp y$
② $\mathsf{S}(x) \div^\sharp \mathsf{S}(y) \to (x - y) \div^\sharp \mathsf{S}(y)$
③ $\mathsf{S}(x) \div^\sharp \mathsf{S}(y) \to x -^\sharp y$
④ $\mathsf{S}(x) +^\sharp y \to x +^\sharp y$
⑤ $(x - y) -^\sharp z \to x -^\sharp (y + z)$
⑥ $(x - y) -^\sharp z \to y +^\sharp z$

dependency graph



5 cycles {①}, {②}, {④}, {⑤}, {①, ⑤} subterm criterion applies

Theorem

if $\exists$ argument filtering π and $\exists$ reduction pair $(\succsim, \succ)$ such that

1. $\pi(\mathcal{R}) \subseteq \succsim$
2. $\pi(\mathcal{C}) \subseteq \succsim \cup \succ$
3. $\pi(\mathcal{C}) \cap \succ \neq \varnothing$

then $\neg\, \exists\, \mathcal{C}$-minimal rewrite sequence

rewrite rules

$$
\begin{array}{rcl@{\qquad}rcl}
0 - y & \to & 0 & 0 + y & \to & y \\
x - 0 & \to & x & \mathsf{S}(x) + y & \to & \mathsf{S}(x + y) \\
\mathsf{S}(x) - \mathsf{S}(y) & \to & x - y & (x - y) - z & \to & x - (y + z) \\
0 \div \mathsf{S}(y) & \to & 0 & \mathsf{S}(x) \div \mathsf{S}(y) & \to & \mathsf{S}((x - y) \div \mathsf{S}(y))
\end{array}
$$

dependency pair ② $\mathsf{S}(x) \div^\sharp \mathsf{S}(y) \to (x - y) \div^\sharp \mathsf{S}(y)$

cycle {②} AF $\pi(-) = 1$ LPO with $\div \sqsupset \mathsf{S}$ $\div^\sharp \sqsupset -^\sharp$