Advanced Topics in Term Rewriting
LVA 703610

`http://cl-informatik.uibk.ac.at/teaching/ws06/attr/`

Georg Moser

office hours: Tuesday, 16:00–18:00 (3M09)

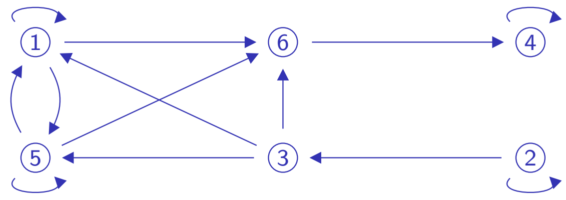
rewrite rules

$$\begin{array}{rlrl} 0 - y &\to 0 & \mathsf{S}(x) \div \mathsf{S}(y) &\to \mathsf{S}((x-y) \div \mathsf{S}(y)) \\ x - 0 &\to x & 0 + y &\to y \\ \mathsf{S}(x) - \mathsf{S}(y) &\to x - y & \mathsf{S}(x) + y &\to \mathsf{S}(x+y) \\ 0 \div \mathsf{S}(y) &\to 0 & (x-y) - z &\to x - (y+z) \end{array}$$

dependency pairs

① $\mathsf{S}(x) -^\sharp \mathsf{S}(y) \to x -^\sharp y$
② $\mathsf{S}(x) \div^\sharp \mathsf{S}(y) \to (x-y) \div^\sharp \mathsf{S}(y)$
③ $\mathsf{S}(x) \div^\sharp \mathsf{S}(y) \to x -^\sharp y$
④ $\mathsf{S}(x) +^\sharp y \to x +^\sharp y$
⑤ $(x-y) -^\sharp z \to x -^\sharp (y+z)$
⑥ $(x-y) -^\sharp z \to y +^\sharp z$

dependency graph



5 cycles {①}, {②}, {④}, {⑤}, {①, ⑤}

Theorem

$\forall$ non-terminating TRS $\mathcal{R}$ $\exists$ cycle $\mathcal{C}$ in $\mathsf{DG}(\mathcal{R})$
$\exists$ $\mathcal{C}$-minimal rewrite sequence $t_1 \to_{\mathcal{R}}^* t_2 \to_{\mathcal{C}} t_3 \to_{\mathcal{R}}^* t_4 \to_{\mathcal{C}} \cdots$

Idea

project each dependency symbol in $\mathcal{C}$ to fixed argument position

$$\pi(t_1) \to_{\mathcal{R}}^* \pi(t_2) \ \ ? \ \ \pi(t_3) \to_{\mathcal{R}}^* \pi(t_4) \ \ ? \ \ \cdots$$

Observation

- ➡ $\pi(t_1)$ is terminating with respect to $\to_{\mathcal{R}}$ (because $t_1 \in \mathcal{T}_\infty^\sharp$)
- ➡ and also with respect to $\to_{\mathcal{R}} \cup \rhd$ (recall $\rhd \cdot \to_{\mathcal{R}} \subseteq \to_{\mathcal{R}} \cdot \rhd$)

Idea

require: $\forall\, l \to r \in \mathcal{C}$ $\pi(l) \trianglerighteq \pi(r)$ and $\exists\, l \to r \in \mathcal{C}$ $\pi(l) \rhd \pi(r)$

Subterm Criterion

Definition

- ➡ simple projection for cycle $\mathcal{C}$ in $\mathsf{DG}(\mathcal{R})$ is mapping π that assigns to every n-ary dependency pair symbol $f^\sharp$ in $\mathcal{C}$ one of its argument positions
- ➡ extension of π to terms in $\mathcal{T}^\sharp$: $\pi(f^\sharp(t_1, \ldots, t_n)) = t_{\pi(f^\sharp)}$

Theorem

if $\exists$ simple projection π for cycle $\mathcal{C}$ in $\mathsf{DG}(\mathcal{R})$ such that

1. $\forall\, l \to r \in \mathcal{C}$ $\pi(l) = \pi(r)$ or $\pi(l) \rhd \pi(r)$ $\pi(\mathcal{C}) \subseteq \trianglerighteq$
2. $\exists\, l \to r \in \mathcal{C}$ $\pi(l) \rhd \pi(r)$ $\pi(\mathcal{C}) \cap \rhd \neq \varnothing$

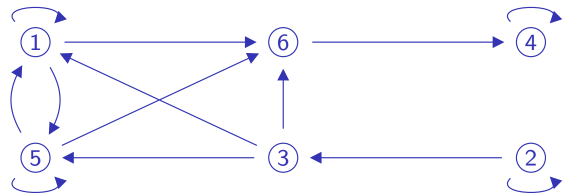
then $\neg\, \exists$ $\mathcal{C}$-minimal rewrite sequence rewrite rules

$$\begin{aligned} 0 - y &\to 0 & \mathsf{S}(x) \div \mathsf{S}(y) &\to \mathsf{S}((x-y) \div \mathsf{S}(y)) \\ x - 0 &\to x & 0 + y &\to y \\ \mathsf{S}(x) - \mathsf{S}(y) &\to x - y & \mathsf{S}(x) + y &\to \mathsf{S}(x+y) \\ 0 \div \mathsf{S}(y) &\to 0 & (x-y) - z &\to x - (y+z) \end{aligned}$$

dependency pairs

① $\mathsf{S}(x) -^\sharp \mathsf{S}(y) \to x -^\sharp y$
② $\mathsf{S}(x) \div^\sharp \mathsf{S}(y) \to (x-y) \div^\sharp \mathsf{S}(y)$
③ $\mathsf{S}(x) \div^\sharp \mathsf{S}(y) \to x -^\sharp y$
④ $\mathsf{S}(x) +^\sharp y \to x +^\sharp y$
⑤ $(x-y) -^\sharp z \to x -^\sharp (y+z)$
⑥ $(x-y) -^\sharp z \to y +^\sharp z$

dependency graph



5 cycles {①}, {②}, {④}, {⑤}, {①, ⑤} subterm criterion applies

Theorem

if $\exists$ argument filtering π and $\exists$ reduction pair $(\succsim, \succ)$ such that

1. $\pi(\mathcal{R}) \subseteq \succsim$
2. $\pi(\mathcal{C}) \subseteq \succsim \cup \succ$
3. $\pi(\mathcal{C}) \cap \succ \neq \varnothing$

then $\neg \exists$ $\mathcal{C}$-minimal rewrite sequence

rewrite rules

$$\begin{aligned} 0 - y &\to 0 & 0 + y &\to y \\ x - 0 &\to x & \mathsf{S}(x) + y &\to \mathsf{S}(x+y) \\ \mathsf{S}(x) - \mathsf{S}(y) &\to x - y & (x-y) - z &\to x - (y+z) \\ 0 \div \mathsf{S}(y) &\to 0 & \mathsf{S}(x) \div \mathsf{S}(y) &\to \mathsf{S}((x-y) \div \mathsf{S}(y)) \end{aligned}$$

dependency pair ② $\mathsf{S}(x) \div^\sharp \mathsf{S}(y) \to (x-y) \div^\sharp \mathsf{S}(y)$

cycle {②} AF $\pi(-) = 1$ LPO with $\div \sqsupset \mathsf{S}$ $\div^\sharp \sqsupset -^\sharp$