Advanced Topics in Term Rewriting
LVA 703610

http://cl-informatik.uibk.ac.at/teaching/ws06/attr/

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Tree Automata

➡ tree automaton is quadruple $\mathcal{A} = (\mathcal{F}, Q, Q_f, \Delta)$ with

① $\mathcal{F}$ signature
② Q states
③ $Q_f \subseteq Q$ final states
④ Δ transition rules

$$\begin{aligned} \mathsf{f}(\alpha_1, \dots, \alpha_n) &\to \beta \\ \alpha &\to \beta \qquad \text{epsilon transition} \end{aligned}$$

➡ language accepted by $\mathcal{A}$:

$$L(\mathcal{A}) = \{ t \in \mathcal{T}(\mathcal{F}) \mid \exists\, \alpha \in Q_f : t \xrightarrow[\Delta]{*} \alpha \}$$

➡ $L \subseteq \mathcal{T}(\mathcal{F})$ is regular if $L = L(\mathcal{A})$ for some tree automaton $\mathcal{A}$

tree automaton $\mathcal{A}$

- signature: a (constant) f g (unary)
- states: α $\underline{\beta}$ final
- transition rules:

$$\mathsf{a} \to \alpha \qquad \mathsf{g}(\alpha) \to \alpha$$
$$\mathsf{f}(\alpha) \to \beta \qquad \beta \to \alpha$$

$$\mathsf{f}(\mathsf{f}(\mathsf{a})) \in L(\mathcal{A}): \quad \mathsf{f}(\mathsf{f}(\mathsf{a})) \to \mathsf{f}(\mathsf{f}(\alpha)) \to \mathsf{f}(\beta) \to \mathsf{f}(\alpha) \to \beta$$
$$\mathsf{g}(\mathsf{a}) \notin L(\mathcal{A}): \quad \mathsf{g}(\mathsf{a}) \to \mathsf{g}(\alpha) \to \alpha$$

$$L(\mathcal{A}) = \{ t \mid \mathrm{root}(t) = \mathsf{f} \}$$

Examples of regular tree languages

- set of all ground terms
- set of well-typed terms with respect to order-sorted signature

sorts: $\mathtt{nat} < \mathtt{int} \qquad \mathtt{bool}$

signature:

$$\mathsf{0} : \mathtt{nat} \qquad + : \mathtt{nat} \times \mathtt{nat} \to \mathtt{nat}$$
$$\mathsf{s} : \mathtt{nat} \to \mathtt{nat} \qquad + : \mathtt{int} \times \mathtt{int} \to \mathtt{int}$$
$$\mathsf{s} : \mathtt{int} \to \mathtt{int} \qquad \leqslant : \mathtt{int} \times \mathtt{int} \to \mathtt{bool}$$
$$\mathsf{p} : \mathtt{int} \to \mathtt{int}$$

tree automaton:

$$\mathsf{0} \to \mathtt{nat} \qquad +(\mathtt{nat}, \mathtt{nat}) \to \mathtt{nat}$$
$$\mathsf{s}(\mathtt{nat}) \to \mathtt{nat} \qquad +(\mathtt{int}, \mathtt{int}) \to \mathtt{int}$$
$$\mathsf{s}(\mathtt{int}) \to \mathtt{int} \qquad \leqslant(\mathtt{int}, \mathtt{int}) \to \mathtt{bool}$$
$$\mathsf{p}(\mathtt{int}) \to \mathtt{int} \qquad \mathtt{nat} \to \mathtt{int}$$

Deterministic Tree Automata

Theorem

every regular language is accepted by deterministic completely defined tree automaton $\mathcal{A} = (\mathcal{F}, Q, Q_f, \Delta)$

- ➡ no epsilon transitions
- ➡ no different transition rules with same left-hand sides
- ➡ $\forall f \in \mathcal{F} \;\; \forall \alpha_1, \dots, \alpha_n \in Q \;\; \exists f(\alpha_1, \dots, \alpha_n) \to \beta \in \Delta$

Proof by Example

$\mathcal{A}_1$

$$\begin{aligned} \mathsf{a} &\to \alpha \\ \mathsf{f}(\alpha) &\to \beta \\ \mathsf{g}(\alpha) &\to \alpha \\ \beta &\to \alpha \end{aligned}$$

$\mathcal{A}_2$

$$\begin{aligned} \mathsf{a} &\to \alpha \\ \mathsf{f}(\alpha) &\to \beta \\ \mathsf{g}(\alpha) &\to \alpha \\ \mathsf{f}(\alpha) &\to \alpha \end{aligned}$$

① remove epsilon transitions

$\mathcal{A}_3$

$$\begin{aligned} \mathsf{a} &\to \{\alpha\} \\ \mathsf{f}(\varnothing) &\to \varnothing \\ \mathsf{f}(\{\alpha\}) &\to \{\alpha, \beta\} \\ \mathsf{f}(\{\beta\}) &\to \varnothing \\ \mathsf{f}(\{\alpha, \beta\}) &\to \{\alpha, \beta\} \end{aligned} \qquad \begin{aligned} \mathsf{g}(\varnothing) &\to \varnothing \\ \mathsf{g}(\{\alpha\}) &\to \{\alpha\} \\ \mathsf{g}(\{\beta\}) &\to \varnothing \\ \mathsf{g}(\{\alpha, \beta\}) &\to \{\alpha\} \end{aligned}$$

② subset construction

③ remove inaccessible states (optional)

Closure Properties

Theorem

regular languages are effectively closed under union, intersection and difference

Proof

union

tree automaton $\mathcal{A}_1 = (\mathcal{F}, Q_1, Q_f^1, \Delta_1)$

tree automaton $\mathcal{A}_2 = (\mathcal{F}, Q_2, Q_f^2, \Delta_2)$

wlog $Q_1 \cap Q_2 = \varnothing$

$L(\mathcal{A}_1) \cup L(\mathcal{A}_2) = L(\mathcal{A})$ for tree automaton $\mathcal{A} = (\mathcal{F}, Q, Q_f, \Delta)$ with

- $Q = Q_1 \cup Q_2$
- $Q_f = Q_f^1 \cup Q_f^2$
- $\Delta = \Delta_1 \cup \Delta_2$

□

Proof

intersection

tree automaton $\mathcal{A}_1 = (\mathcal{F}, Q_1, Q_f^1, \Delta_1)$
tree automaton $\mathcal{A}_2 = (\mathcal{F}, Q_2, Q_f^2, \Delta_2)$

$L(\mathcal{A}_1) \cap L(\mathcal{A}_2) = L(\mathcal{A})$ for tree automaton $\mathcal{A} = (\mathcal{F}, Q, Q_f, \Delta)$ with

- $Q = Q_1 \times Q_2$
- $Q_f = Q_f^1 \times Q_f^2$
- Δ:

$$\mathsf{f}([\alpha_1, \beta_1], \ldots, [\alpha_n, \beta_n]) \to [\alpha, \beta] \quad \forall\ \mathsf{f}(\alpha_1, \ldots, \alpha_n) \to \alpha \in \Delta_1 \quad \forall\ \mathsf{f}(\beta_1, \ldots, \beta_n) \to \beta \in \Delta_2$$

$$[\alpha, \beta] \to [\alpha', \beta] \quad \forall\ \alpha \to \alpha' \in \Delta_1$$

$$[\alpha, \beta] \to [\alpha, \beta'] \quad \forall\ \beta \to \beta' \in \Delta_2$$

□

Proof

difference

tree automaton $\mathcal{A}_1 = (\mathcal{F}, Q_1, Q_f^1, \Delta_1)$
tree automaton $\mathcal{A}_2 = (\mathcal{F}, Q_2, Q_f^2, \Delta_2)$

deterministic completely defined

$L(\mathcal{A}_1) \setminus L(\mathcal{A}_2) = L(\mathcal{A})$ for tree automaton $\mathcal{A} = (\mathcal{F}, Q, Q_f, \Delta)$ with

- ➡ $Q \;=\; Q_1 \times Q_2$
- ➡ $Q_f \;=\; Q_f^1 \times (Q_2 \setminus Q_f^2)$
- ➡ Δ:

$$\mathsf{f}([\alpha_1, \beta_1], \ldots, [\alpha_n, \beta_n]) \to [\alpha, \beta] \quad \forall\, \mathsf{f}(\alpha_1, \ldots, \alpha_n) \to \alpha \in \Delta_1, \; \forall\, \mathsf{f}(\beta_1, \ldots, \beta_n) \to \beta \in \Delta_2$$

$$[\alpha, \beta] \to [\alpha', \beta] \quad \forall\, \alpha \to \alpha' \in \Delta_1$$

□

Decidability

Theorem

➡ membership	instance:	tree automaton $\mathcal{A}$ and term t
	question:	$t \in L(\mathcal{A})$?
➡ emptiness	instance:	tree automaton $\mathcal{A}$
	question:	$L(\mathcal{A}) = \varnothing$?
➡ finiteness	instance:	tree automaton $\mathcal{A}$
	question:	$L(\mathcal{A})$ is finite?
➡ inclusion	instance:	tree automata $\mathcal{A}_1$ and $\mathcal{A}_2$
	question:	$L(\mathcal{A}_1) \subseteq L(\mathcal{A}_2)$?

are decidable problems