Advanced Topics in Term Rewriting
LVA 703610

http://cl-informatik.uibk.ac.at/teaching/ws06/attr/

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office hours: Tuesday, 16:00–18:00 (3M09)

Pumping Lemma

$$L \text{ is regular} \implies \begin{cases} \exists\ n \geqslant 1 & \\ \forall\ t \in L & \text{with } \mathrm{depth}(t) \geqslant n \\ \exists\ C_1, C_2, t' & \text{with } \begin{cases} t = C_1[C_2[t']] \\ C_2 \neq \Box \end{cases} \\ \forall\ i \geqslant 0 & C_1[\underbrace{C_2[\cdots C_2}_{i}[t']\cdots]] \in L \end{cases}$$

Proof Sketch

n: number of states of any tree automaton that accepts L □

Corollary

If $L(\mathcal{A}) \neq \varnothing$ then $\exists t \in L(\mathcal{A})$, $\mathrm{depth}(t) <$ number of states in $\mathcal{A}$

Decidability

Theorem

- membership
  instance: tree automaton $\mathcal{A}$ and term t
  question: $t \in L(\mathcal{A})$?
- emptiness $\iff \neg\,\exists\, t \in L(\mathcal{A})\colon \mathsf{depth}(t) < |Q|$
  instance: tree automaton $\mathcal{A}$
  question: $L(\mathcal{A}) = \varnothing$?
- finiteness $\iff \neg\,\exists\, t \in L(\mathcal{A})\colon |Q| \leqslant \mathsf{depth}(t) < 2|Q|$
  instance: tree automaton $\mathcal{A}$
  question: $L(\mathcal{A})$ is finite?
- inclusion
  instance: tree automata $\mathcal{A}_1$ and $\mathcal{A}_2$
  question: $L(\mathcal{A}_1) \subseteq L(\mathcal{A}_2)$?

are decidable problems

Words and Terms

Definition

1. $\mathsf{yield}(L) = \{\, \mathsf{yield}(t) \mid t \in L \,\}$
2. $$\mathsf{yield}(t) = \begin{cases} \mathsf{a} & \text{if } t = \mathsf{a} \\ \mathsf{yield}(t_1) \cdots \mathsf{yield}(t_n) & \text{if } t = \mathsf{f}(t_1, \dots, t_n) \end{cases}$$

Theorem

1. $\forall$ context-free word grammar G
   set of parse trees of G is regular tree language
2. $\forall$ regular tree language L
   $\mathsf{yield}(L)$ is context-free word language

Ground Tree Transducers

➡ ground tree transducer is pair $\mathcal{G} = (\mathcal{A}, \mathcal{B})$ of tree automata

$$\begin{aligned} \mathcal{A} &= (\mathcal{F}, Q_A, —, \Delta_A) \\ \mathcal{B} &= (\mathcal{F}, Q_B, —, \Delta_B) \end{aligned}$$

➡ relation accepted by $\mathcal{G}$:

$$R(\mathcal{G}) = \{ (s,t) \mid s \xrightarrow[\Delta_A]{*} \cdot \xleftarrow[\Delta_B]{*} t \}$$

➡ $R \subseteq \mathcal{T}(\mathcal{F}) \times \mathcal{T}(\mathcal{F})$ is recognisable if $R = R(\mathcal{G})$ for some ground tree transducer $\mathcal{G}$

Theorem

$\mathrel{+\!\!\!+\!\!\!\!\to}_{\mathcal{R}}$ is recognisable for finite left-linear right-ground $\mathcal{R}$

Proof by Example

$$\mathcal{R} \quad \begin{aligned} 0 + y &\to \mathsf{S}(0) \\ \mathsf{S}(x) + y &\to \mathsf{S}(0+0) \\ 0 \times y &\to 0 \\ \mathsf{S}(x) \times y &\to 0 + 0 \end{aligned}$$

$$\mathrel{+\!\!\!+\!\!\!\!\to}_{\mathcal{R}} = R(\mathcal{G}) \qquad \mathcal{G} = (\mathcal{A}, \mathcal{B})$$

$0 \to *$				0	$\leftarrow$	0
$\mathsf{S}(*) \to *$				$0+0$	$\leftarrow$	$0+0$
$* + * \to *$	$0 + *$	$\to$	1	1	$\leftarrow$	$\mathsf{S}(0)$
$* \times * \to *$	$\mathsf{S}(*) + *$	$\to$	2	2	$\leftarrow$	$\mathsf{S}(0+0)$
$0 \to 0$	$0 \times *$	$\to$	3	3	$\leftarrow$	0
$\mathsf{S}(*) \to \mathsf{S}(*)$	$\mathsf{S}(*) \times *$	$\to$	4	4	$\leftarrow$	$0+0$

$\square$

Properties of Recognisable Relations

- (effectively) closed under
  1. inverse
  2. composition
  3. transitive closure

- inclusion

  instance: ground tree transducers $\mathcal{G}_1$, $\mathcal{G}_2$
  question: $L(\mathcal{G}_1) \subseteq L(\mathcal{G}_2)$?

  is decidable

- $\forall$ recognisable $R \subseteq \mathcal{T}(\mathcal{F}) \times \mathcal{T}(\mathcal{F})$ $\forall$ regular $L \subseteq \mathcal{T}(\mathcal{F})$

$$R[L] = \{ t \mid \exists\, s \in L \colon (s,t) \in R \}$$

  is regular

Theorem

recognisable relations are closed under inverse

Proof

ground tree transducer $\mathcal{G} = (\mathcal{F}, Q, \Delta_A, \Delta_B)$
$R(\mathcal{G})^{-1}$ is accepted by $(\mathcal{F}, Q, \Delta_B, \Delta_A)$

□

Lemma

$\forall$ ground tree transducer $\mathcal{G} = (\mathcal{F}, Q, \Delta_A, \Delta_B)$

set of transition rules

$$\Delta_\epsilon(\mathcal{A}, \mathcal{B}) = \left\{ \alpha \to \beta \;\middle|\; \exists\, t \in \mathcal{T}(\mathcal{F}) \colon \alpha \xleftarrow[\Delta_A]{*} t \xrightarrow[\Delta_B]{*} \beta \right\}$$

is computable

Theorem

recognisable relations are closed under composition

Proof

ground tree transducer $\mathcal{G}_1 = (\mathcal{F}, Q_1, \Delta_{A_1}, \Delta_{B_1})$
ground tree transducer $\mathcal{G}_2 = (\mathcal{F}, Q_2, \Delta_{A_2}, \Delta_{B_2})$

wlog $Q_1 \cap Q_2 = \varnothing$

$R(\mathcal{G}_1) \cdot R(\mathcal{G}_2)$ is accepted by $\mathcal{G} = (\mathcal{F}, Q, \Delta_A, \Delta_B)$ with

- $Q = Q_1 \cup Q_2$
- $\Delta_A = \Delta_{A_1} \cup \Delta_{A_2} \cup \Delta_\epsilon(\mathcal{B}_1, \mathcal{A}_2)$
- $\Delta_B = \Delta_{B_1} \cup \Delta_{B_2} \cup \Delta_\epsilon(\mathcal{A}_2, \mathcal{B}_1)$

□

Pumping Lemma Decidability Ground Tree Transducers Recognisability

Theorem

recognisable relations are closed under transitive closure

Proof

ground tree transducer $\mathcal{G} = (\mathcal{F}, Q, \Delta_A, \Delta_B)$

define $\mathcal{G}_i = (\mathcal{F}, Q, \Delta_{A_i}, \Delta_{B_i})$ with

$$\begin{aligned} \Delta_{A_1} &= \Delta_A & \quad \Delta_{A_{i+1}} &= \Delta_{A_i} \cup \Delta_\epsilon(\mathcal{B}_i, \mathcal{A}_i) \\ \Delta_{B_1} &= \Delta_B & \quad \Delta_{B_{i+1}} &= \Delta_{B_i} \cup \Delta_\epsilon(\mathcal{A}_i, \mathcal{B}_i) \end{aligned}$$

$R(\mathcal{G})^+$ is accepted by $\mathcal{G}_n$ for n with $\mathcal{G}_n = \mathcal{G}_{n+1}$

□

Theorem

$\forall$ recognisable $R \subseteq \mathcal{T}(\mathcal{F}) \times \mathcal{T}(\mathcal{F})$ $\forall$ regular $L \subseteq \mathcal{T}(\mathcal{F})$
$R[L] = \{\, t \in \mathcal{T}(\mathcal{F}) \mid \exists s \in L \colon s\ R\ t \,\}$ is regular

Proof

$R = R(\mathcal{G})$ for ground tree transducer $\mathcal{G} = (\mathcal{F}, Q, \Delta_A, \Delta_B)$
$L = L(\mathcal{C})$ for tree automaton $\mathcal{C} = (\mathcal{F}, Q', Q_f, \Delta_C)$

wlog $Q \cap Q' = \varnothing$

$(\mathcal{F}, Q \cup Q', Q_f, \Delta)$ with

$$\Delta = \Delta_B \cup \Delta_C \cup \Delta_\epsilon(\mathcal{A}, \mathcal{C})$$

$\square$

Lemma

$\forall$ recognisable $R \subseteq \mathcal{T}(\mathcal{F}) \times \mathcal{T}(\mathcal{F})$

$$\varphi(R) = \{\, (s \otimes t){\downarrow_{\mathcal{S}}} \mid s\ R\ t \,\}$$

with binary $\otimes \notin \mathcal{F}$ and TRS $\mathcal{S}$ consisting of all rules

$$f(x_1, \dots, x_n) \otimes f(y_1, \dots, y_n) \rightarrow f(x_1 \otimes y_1, \dots, x_n \otimes y_n)$$

is regular

Theorem

$$R_1 \subseteq R_2 \iff \varphi(R_1) \subseteq \varphi(R_2)$$