

Advanced Topics in Term Rewriting

LVA 703610

<http://cl-informatik.uibk.ac.at/teaching/ws06/attr/>

Georg Moser

office hours: **Tuesday, 16:00–18:00** (3M09)

Decidability

Theorem

→ membership

instance: tree automaton \mathcal{A} and term t

question: $t \in L(\mathcal{A})$?

→ emptiness $\iff \neg \exists t \in L(\mathcal{A}) : \text{depth}(t) < |\mathcal{Q}|$

instance: tree automaton \mathcal{A}

question: $L(\mathcal{A}) = \emptyset$?

→ finiteness $\iff \neg \exists t \in L(\mathcal{A}) : |\mathcal{Q}| \leq \text{depth}(t) < 2|\mathcal{Q}|$

instance: tree automaton \mathcal{A}

question: $L(\mathcal{A})$ is finite?

→ inclusion

instance: tree automata \mathcal{A}_1 and \mathcal{A}_2

question: $L(\mathcal{A}_1) \subseteq L(\mathcal{A}_2)$?

are **decidable** problems

Pumping Lemma

$$L \text{ is regular} \implies \left\{ \begin{array}{l} \exists n \geq 1 \\ \forall t \in L \quad \text{with } \text{depth}(t) \geq n \\ \exists C_1, C_2, t' \quad \text{with } \begin{cases} t = C_1[C_2[t']] \\ C_2 \neq \square \end{cases} \\ \forall i \geq 0 \quad C_1[\underbrace{C_2[\dots C_2[t']\dots]}_i] \in L \end{array} \right.$$

Proof Sketch

n : number of states of any tree automaton that accepts L \square

Corollary

If $L(\mathcal{A}) \neq \emptyset$ then $\exists t \in L(\mathcal{A})$, $\text{depth}(t) < \text{number of states in } \mathcal{A}$

Words and Terms

Definition

$$1 \text{ yield}(L) = \{ \text{yield}(t) \mid t \in L \}$$

$$2 \text{ yield}(t) = \begin{cases} a & \text{if } t = a \\ \text{yield}(t_1) \cdots \text{yield}(t_n) & \text{if } t = f(t_1, \dots, t_n) \end{cases}$$

Theorem

1 \forall context-free word grammar G
set of **parse trees** of G is regular tree language

2 \forall regular tree language L
yield(L) is context-free word language

Ground Tree Transducers

→ **ground tree transducer** is pair $\mathcal{G} = (\mathcal{A}, \mathcal{B})$ of tree automata

$$\mathcal{A} = (\mathcal{F}, Q_A, \rightarrow, \Delta_A)$$

$$\mathcal{B} = (\mathcal{F}, Q_B, \leftarrow, \Delta_B)$$

→ relation **accepted** by \mathcal{G} :

$$R(\mathcal{G}) = \{ (s, t) \mid s \xrightarrow{\Delta_A^*} \cdot \xleftarrow{\Delta_B^*} t \}$$

→ $R \subseteq \mathcal{T}(\mathcal{F}) \times \mathcal{T}(\mathcal{F})$ is **recognisable** if $R = R(\mathcal{G})$ for some ground tree transducer \mathcal{G}

Theorem

$\dashv\vdash_{\mathcal{R}}$ is **recognisable** for finite **left-linear right-ground** \mathcal{R}

Proof by Example

$$\mathcal{R} \quad \begin{array}{l} 0 + y \rightarrow S(0) \\ S(x) + y \rightarrow S(0 + 0) \\ 0 \times y \rightarrow 0 \\ S(x) \times y \rightarrow 0 + 0 \end{array}$$

$$\dashv\vdash_{\mathcal{R}} = R(\mathcal{G}) \quad \mathcal{G} = (\mathcal{A}, \mathcal{B})$$

$0 \rightarrow *$	$0 \leftarrow 0$	
$S(*) \rightarrow *$	$0 + 0 \leftarrow 0 + 0$	
$* + * \rightarrow *$	$0 + * \rightarrow 1$	$1 \leftarrow S(0)$
$* \times * \rightarrow *$	$S(*) + * \rightarrow 2$	$2 \leftarrow S(0 + 0)$
$0 \rightarrow 0$	$0 \times * \rightarrow 3$	$3 \leftarrow 0$
$S(*) \rightarrow S(*)$	$S(*) \times * \rightarrow 4$	$4 \leftarrow 0 + 0$

□

Properties of Recognisable Relations

→ (effectively) **closed under**

- 1 inverse
- 2 composition
- 3 transitive closure

→ **inclusion**

instance: ground tree transducers $\mathcal{G}_1, \mathcal{G}_2$

question: $L(\mathcal{G}_1) \subseteq L(\mathcal{G}_2)$?

is **decidable**

→ \forall recognisable $R \subseteq \mathcal{T}(\mathcal{F}) \times \mathcal{T}(\mathcal{F}) \quad \forall$ regular $L \subseteq \mathcal{T}(\mathcal{F})$

$$R[L] = \{ t \mid \exists s \in L : (s, t) \in R \}$$

is **regular**

Theorem

recognisable relations are closed under **inverse**

Proof

ground tree transducer $\mathcal{G} = (\mathcal{F}, Q, \Delta_A, \Delta_B)$

$R(\mathcal{G})^{-1}$ is accepted by $(\mathcal{F}, Q, \Delta_B, \Delta_A)$

□

Lemma

\forall ground tree transducer $\mathcal{G} = (\mathcal{F}, Q, \Delta_A, \Delta_B)$

set of transition rules

$$\Delta_\epsilon(\mathcal{A}, \mathcal{B}) = \left\{ \alpha \rightarrow \beta \mid \exists t \in \mathcal{T}(\mathcal{F}) : \alpha \xleftarrow{\Delta_A^*} t \xrightarrow{\Delta_B^*} \beta \right\}$$

is **computable**

Theorem

recognisable relations are closed under **composition**

Proof

ground tree transducer $\mathcal{G}_1 = (\mathcal{F}, Q_1, \Delta_{A_1}, \Delta_{B_1})$

ground tree transducer $\mathcal{G}_2 = (\mathcal{F}, Q_2, \Delta_{A_2}, \Delta_{B_2})$

wlog $Q_1 \cap Q_2 = \emptyset$

$R(\mathcal{G}_1) \cdot R(\mathcal{G}_2)$ is accepted by $\mathcal{G} = (\mathcal{F}, Q, \Delta_A, \Delta_B)$ with

- $Q = Q_1 \cup Q_2$
- $\Delta_A = \Delta_{A_1} \cup \Delta_{A_2} \cup \Delta_\epsilon(\mathcal{B}_1, \mathcal{A}_2)$
- $\Delta_B = \Delta_{B_1} \cup \Delta_{B_2} \cup \Delta_\epsilon(\mathcal{A}_2, \mathcal{B}_1)$

□

Theorem

recognisable relations are closed under **transitive closure**

Proof

ground tree transducer $\mathcal{G} = (\mathcal{F}, Q, \Delta_A, \Delta_B)$

define $\mathcal{G}_i = (\mathcal{F}, Q, \Delta_{A_i}, \Delta_{B_i})$ with

$$\begin{aligned} \Delta_{A_1} &= \Delta_A & \Delta_{A_{i+1}} &= \Delta_{A_i} \cup \Delta_\epsilon(\mathcal{B}_i, \mathcal{A}_i) \\ \Delta_{B_1} &= \Delta_B & \Delta_{B_{i+1}} &= \Delta_{B_i} \cup \Delta_\epsilon(\mathcal{A}_i, \mathcal{B}_i) \end{aligned}$$

$R(\mathcal{G})^+$ is accepted by \mathcal{G}_n for n with $\mathcal{G}_n = \mathcal{G}_{n+1}$

□

Theorem

\forall recognisable $R \subseteq T(\mathcal{F}) \times T(\mathcal{F})$ \forall regular $L \subseteq T(\mathcal{F})$

$R[L] = \{t \in T(\mathcal{F}) \mid \exists s \in L: s R t\}$ is regular

Proof

$R = R(\mathcal{G})$ for ground tree transducer $\mathcal{G} = (\mathcal{F}, Q, \Delta_A, \Delta_B)$

$L = L(\mathcal{C})$ for tree automaton $\mathcal{C} = (\mathcal{F}, Q', Q_f, \Delta_C)$

wlog $Q \cap Q' = \emptyset$

$(\mathcal{F}, Q \cup Q', Q_f, \Delta)$ with

$$\Delta = \Delta_B \cup \Delta_C \cup \Delta_\epsilon(\mathcal{A}, \mathcal{C})$$

□

Lemma

\forall recognisable $R \subseteq T(\mathcal{F}) \times T(\mathcal{F})$

$$\varphi(R) = \{(s \otimes t) \downarrow_{\mathcal{S}} \mid s R t\}$$

with binary $\otimes \notin \mathcal{F}$ and TRS \mathcal{S} consisting of all rules

$$f(x_1, \dots, x_n) \otimes f(y_1, \dots, y_n) \rightarrow f(x_1 \otimes y_1, \dots, x_n \otimes y_n)$$

is **regular**

Theorem

$$R_1 \subseteq R_2 \iff \varphi(R_1) \subseteq \varphi(R_2)$$