Functional Programming

1 (a) *explanation*

$$\begin{aligned} \texttt{s (s 10)} &\to \texttt{s (10 * 10)} \\ &\to \texttt{s 100} \\ &\to \texttt{100 * 100} \\ &\to \texttt{10000} \end{aligned}$$

(b) *explanation*

$$\begin{aligned} \texttt{s (s 10)} &\to \texttt{(s 10) * (s 10)} \\ &\to \texttt{(10 * 10) * (s 10)} \\ &\to \texttt{100 * (s 10)} \\ &\to \texttt{100 * (10 * 10)} \\ &\to \texttt{100 * 100} \\ &\to \texttt{10000} \end{aligned}$$

2 (a) *base case*

Let $t = \texttt{E}$. The base case concludes by the derivation

$$\begin{aligned} \texttt{mirror}\ (\texttt{mirror}\ t) &= \texttt{mirror}\ (\texttt{mirror E}) && (\text{since } t = \texttt{E}) \\ &= \texttt{mirror E} && (\text{by definition of } \texttt{mirror}) \\ &= \texttt{E} && (\text{by definition of } \texttt{mirror}) \\ &= t. \end{aligned}$$

(b) *step case*

Let $t = \texttt{N}\ (t_1, t_2)$. By IH it holds that

$$\begin{aligned} \texttt{mirror}\ (\texttt{mirror}\ t_1) &= t_1 \\ \texttt{mirror}\ (\texttt{mirror}\ t_2) &= t_2. \end{aligned}$$

The step case concludes by the derivation

$$\begin{aligned} \texttt{mirror}\ (\texttt{mirror}\ t) &= \texttt{mirror}\ (\texttt{mirror}\ (\texttt{N}\ (t_1, t_2))) && (t = \texttt{N}\ (t_1, t_2)) \\ &= \texttt{mirror}\ (\texttt{N}\ (\texttt{mirror}\ t_2, \texttt{mirror}\ t_1)) && (\text{def. of } \texttt{mirror}) \\ &= \texttt{N}\ (\texttt{mirror}\ (\texttt{mirror}\ t_1), \texttt{mirror}\ (\texttt{mirror}\ t_2)) && (\text{def. of } \texttt{mirror}) \\ &= \texttt{N}\ (t_1, t_2) && (\text{by IH}) \\ &= t. \end{aligned}$$

3 (a) *implementation*

```
let f x =
 let rec f x acc =
  if x / 2 = 0 then acc else f (x / 2) (acc + 1)
 in f x 0
;;
```

(b) *implementation*

```
let g x =
 let rec g x =
  if x < 2 then (1, 1) else
   let (g1, g2) = g (x - 1) in (g2, g2 + 2 * g1)
 in snd (g x)
;;
```

4 (a) *explanation*

$$\begin{aligned}(\lambda x.y\ x)\ (\lambda y.(\lambda y.y)\ z) &\rightarrow_\beta (\lambda x.y\ x)\ (\lambda y.z)\\ &\rightarrow_\beta y\ (\lambda y.z)\end{aligned}$$

(b) *explanation*

$\mathcal{FV}\mathsf{ar} = \{y, z\}$

(c) *explanation*

$\mathcal{BV}\mathsf{ar} = \{x, y\}$

(d) *explanation*

$\mathcal{S}\mathsf{ub} = \{y, x, y\ x, \lambda x.y\ x, \lambda y.y, z, (\lambda y.y)\ z, \lambda y.(\lambda y.y)\ z, (\lambda x.y\ x)\ (\lambda y.(\lambda y.y)\ z)\}$

5 (a) *explanation*

$$\cfrac{\cfrac{E \vdash \mathsf{tl} : \mathsf{list(int)} \to \mathsf{list(int)} \qquad \star}{E \vdash \mathsf{tl}\ (\mathsf{cons}\ 1\ (\mathsf{cons}\ 2\ \mathsf{nil})) : \mathsf{list(int)}}\,(\text{app}) \qquad \cfrac{E, x : \mathsf{list(int)} \vdash \mathsf{hd} : \mathsf{list(int)} \to \mathsf{int} \qquad E, x : \mathsf{list(int)} \vdash x : \mathsf{list(int)}}{E, x : \mathsf{list(int)} \vdash \mathsf{hd}\ x : \mathsf{int}}\,(\text{app})}{E \vdash \mathbf{let}\ x = \mathsf{tl}\ (\mathsf{cons}\ 1\ (\mathsf{cons}\ 2\ \mathsf{nil}))\ \mathbf{in}\ \mathsf{hd}\ x : \mathsf{int}}\,(\text{let})$$

Where $\star$ is:

$$\cfrac{\cfrac{E \vdash \mathsf{cons} : \mathsf{int} \to \mathsf{list(int)} \to \mathsf{list(int)} \qquad E \vdash 1 : \mathsf{int}}{E \vdash \mathsf{cons}\ 1 : \mathsf{list(int)} \to \mathsf{list(int)}}\,(\text{app}) \qquad \cfrac{\cfrac{E \vdash \mathsf{cons} : \mathsf{int} \to \mathsf{list(int)} \to \mathsf{list(int)} \qquad E \vdash 2 : \mathsf{int}}{E \vdash \mathsf{cons}\ 2 : \mathsf{list(int)} \to \mathsf{list(int)}}\,(\text{app}) \qquad E \vdash \mathsf{nil} : \mathsf{list(int)}}{E \vdash \mathsf{cons}\ 2\ \mathsf{nil} : \mathsf{list(int)}}\,(\text{app})}{E \vdash \mathsf{cons}\ 1\ (\mathsf{cons}\ 2\ \mathsf{nil}) : \mathsf{list(int)}}\,(\text{app})$$

(b) *explanation*

$$\alpha_3 \to \mathsf{list}(\alpha_3) \to \mathsf{list}(\alpha_3) \approx \alpha_2 \to \alpha_1 \to \alpha_4; \mathsf{bool} \approx \alpha_2; \mathsf{list}(\alpha_0) \approx \alpha_1; \mathsf{list}(\alpha_0) \approx \alpha_4$$
$$\Rightarrow_{\iota}^{(\mathsf{d}_2)}$$
$$\alpha_3 \approx \alpha_2; \mathsf{list}(\alpha_3) \to \mathsf{list}(\alpha_3) \approx \alpha_1 \to \alpha_4; \mathsf{bool} \approx \alpha_2; \mathsf{list}(\alpha_0) \approx \alpha_1; \mathsf{list}(\alpha_0) \approx \alpha_4$$
$$\Rightarrow_{\{\alpha_3 \mapsto \alpha_2\}}^{(\mathsf{v}_1)}$$
$$\mathsf{list}(\alpha_2) \to \mathsf{list}(\alpha_2) \approx \alpha_1 \to \alpha_4; \mathsf{bool} \approx \alpha_2; \mathsf{list}(\alpha_0) \approx \alpha_1; \mathsf{list}(\alpha_0) \approx \alpha_4$$
$$\Rightarrow_{\iota}^{(\mathsf{d}_2)}$$
$$\mathsf{list}(\alpha_2) \approx \alpha_1; \mathsf{list}(\alpha_2) \approx \alpha_4; \mathsf{bool} \approx \alpha_2; \mathsf{list}(\alpha_0) \approx \alpha_1; \mathsf{list}(\alpha_0) \approx \alpha_4$$
$$\Rightarrow_{\{\alpha_1 \mapsto \mathsf{list}(\alpha_2)\}}^{(\mathsf{v}_2)}$$
$$\mathsf{list}(\alpha_2) \approx \alpha_4; \mathsf{bool} \approx \alpha_2; \mathsf{list}(\alpha_0) \approx \mathsf{list}(\alpha_2); \mathsf{list}(\alpha_0) \approx \alpha_4$$
$$\Rightarrow_{\{\alpha_4 \mapsto \mathsf{list}(\alpha_2)\}}^{(\mathsf{v}_2)}$$
$$\mathsf{bool} \approx \alpha_2; \mathsf{list}(\alpha_0) \approx \mathsf{list}(\alpha_2); \mathsf{list}(\alpha_0) \approx \mathsf{list}(\alpha_2)$$
$$\Rightarrow_{\{\alpha_2 \mapsto \mathsf{bool}\}}^{(\mathsf{v}_2)}$$
$$\mathsf{list}(\alpha_0) \approx \mathsf{list}(\mathsf{bool}); \mathsf{list}(\alpha_0) \approx \mathsf{list}(\mathsf{bool})$$
$$\Rightarrow_{\iota}^{(\mathsf{d}_1)}$$
$$\alpha_0 \approx \mathsf{bool}; \mathsf{list}(\alpha_0) \approx \mathsf{list}(\mathsf{bool})$$
$$\Rightarrow_{\{\alpha_0 \mapsto \mathsf{bool}\}}^{(\mathsf{v}_1)}$$
$$\mathsf{list}(\mathsf{bool}) \approx \mathsf{list}(\mathsf{bool})$$
$$\Rightarrow_{\iota}^{(\mathsf{t})}$$
$$\square$$

The solution is

$$\{\alpha_3 \mapsto \alpha_2\}\{\alpha_1 \mapsto \mathsf{list}(\alpha_2)\}\{\alpha_4 \mapsto \mathsf{list}(\alpha_2)\}\{\alpha_2 \mapsto \mathsf{bool}\}\{\alpha_0 \mapsto \mathsf{bool}\}$$

which is equivalent to

$$\{\alpha_0 \mapsto \mathsf{bool}, \alpha_1 \mapsto \mathsf{list}(\mathsf{bool}), \alpha_2 \mapsto \mathsf{bool}, \alpha_3 \mapsto \mathsf{bool}, \alpha_4 \mapsto \mathsf{list}(\mathsf{bool})\}.$$