Functional Programming

1 (a) *explanation*

$$\begin{aligned}(\lambda xyz.x\ z\ (y\ z))\ (\lambda xy.x)\ (\lambda x.x)\ (\lambda x.x) &\to_\beta (\lambda yz.(\lambda xy.x)\ z\ (y\ z))\ (\lambda x.x)\ (\lambda x.x)\\ &\to_\beta (\lambda yz.(\lambda y.z)\ (y\ z))\ (\lambda x.x)\ (\lambda x.x)\\ &\to_\beta (\lambda yz.z)\ (\lambda x.x)\ (\lambda x.x)\\ &\to_\beta (\lambda z.z)\ (\lambda x.x)\\ &\to_\beta \lambda x.x\end{aligned}$$

(b) *explanation*

$$\begin{aligned}(\lambda xyz.x\ z\ (y\ z))\ (\lambda xy.x)\ (\lambda x.x)\ (\lambda x.x) &\to_\beta (\lambda yz.(\lambda xy.x)\ z\ (y\ z))\ (\lambda x.x)\ (\lambda x.x)\\ &\to_\beta (\lambda z.(\lambda xy.x)\ z\ ((\lambda x.x)\ z))\ (\lambda x.x)\\ &\to_\beta (\lambda xy.x)\ (\lambda x.x)\ ((\lambda x.x)\ (\lambda x.x))\\ &\to_\beta (\lambda y.(\lambda x.x))\ ((\lambda x.x)\ (\lambda x.x))\\ &\to_\beta \lambda x.x\end{aligned}$$

2 (a) *base case*

Let $t = \texttt{E}$. The base case concludes by the derivation

$$\begin{aligned}\texttt{sum}\ (\texttt{preorder}\ t) &= \texttt{sum}\ (\texttt{preorder E}) && (\text{since } t = \texttt{E})\\ &= \texttt{sum}\ (\texttt{[]}) && (\text{by definition of } \texttt{preorder})\\ &= 0 && (\text{by definition of } \texttt{sum})\\ &= \texttt{sum\_tree E} && (\text{by definition of } \texttt{sum\_tree})\\ &= \texttt{sum\_tree}\ t\end{aligned}$$

(b) *step case*

Let $t = \texttt{N}\ (t_1, v, t_2)$. By IH it holds that

$$\begin{aligned}\texttt{sum}\ (\texttt{preorder}\ t_1) &= \texttt{sum\_tree}\ t_1\\ \texttt{sum}\ (\texttt{preorder}\ t_2) &= \texttt{sum\_tree}\ t_2.\end{aligned}$$

The step case concludes by the derivation

$$\begin{aligned}\texttt{sum}\ (\texttt{preorder}\ t) &= \texttt{sum}\ (\texttt{preorder}\ (\texttt{N}\ (t_1, v, t_2))) && (t = \texttt{N}\ (t_1, v, t_2))\\ &= \texttt{sum}\ (v :: (\texttt{preorder}\ t_1\ \texttt{@}\ \texttt{preorder}\ t_2)) && (\text{def. of } \texttt{preorder})\\ &= v + (\texttt{sum}\ (\texttt{preorder}\ t_1\ \texttt{@}\ \texttt{preorder}\ t_2)) && (\text{def. of } \texttt{sum})\\ &= v + (\texttt{sum}\ (\texttt{preorder}\ t_1) + \texttt{sum}\ (\texttt{preorder}\ t_2)) && (\text{by } (\star))\\ &= v + (\texttt{sum\_tree}\ t_1 + \texttt{sum\_tree}\ t_2) && (\text{by IH})\\ &= \texttt{sum\_tree}\ (\texttt{N}\ (t_1, v, t_2)) && (\text{def. of } \texttt{sum\_tree})\\ &= \texttt{sum\_tree}\ t\end{aligned}$$

3 (a) *implementation*

```
let unique xs =
 let rec rev acc = function
  | [] -> acc
  | x :: xs -> rev (x :: acc) xs
 in
 let rec unique acc = function
  | [] -> rev [] acc
  | x :: xs ->
   if mem x xs then unique acc xs else unique (x :: acc) xs
 in
 unique [] xs
;;
```

(b) *implementation*

```
let percentage x ys =
 let rec p x = function
  | [] -> (0, 0)
  | y :: ys ->
   let (i, j) = p x ys in
   if x = y then (i + 1, j + 1) else (i, j + 1)
 in
 if ys = [] then 0.0 else let (i, j) = p x ys in
 float_of_int i /.  float_of_int j
;;
```

4 (a) *explanation*

$t \rightarrow_\beta y\ (\lambda yz.z\ y)\ w$

(b) *explanation*

$\mathcal{FV}\mathsf{ar}(t) = \{w, y\}$

(c) *explanation*

$\mathcal{BV}\mathsf{ar}(t) = \{x, y, z\}$

(d) *explanation*

$\mathcal{S}\mathsf{ub}(t) = \{y, x, y\ x, \lambda x.y\ x, z, z\ y, \lambda z.z\ y, \lambda yz.z\ y, (\lambda x.y\ x)\ (\lambda yz.z\ y), w, t\}$

5 (a) *explanation*

$$\dfrac{E \vdash 1 : \mathsf{int} \qquad \dfrac{\dfrac{E, x : \mathsf{int} \vdash \mathsf{p} : \mathsf{int} \to \mathsf{int} \to \mathsf{pair}(\mathsf{int}, \mathsf{int}) \qquad E, x : \mathsf{int} \vdash x : \mathsf{int}}{E, x : \mathsf{int} \vdash \mathsf{p}\ \ x : \mathsf{int} \to \mathsf{pair}(\mathsf{int}, \mathsf{int})}\,(\text{app}) \qquad \star}{E, x : \mathsf{int} \vdash \mathsf{p}\ \ x\ (x + x) : \mathsf{pair}(\mathsf{int}, \mathsf{int})}\,(\text{app})}{E \vdash \mathbf{let}\ x = 1\ \mathbf{in}\ \mathsf{p}\ \ x\ (x + x) : \mathsf{pair}(\mathsf{int}, \mathsf{int})}\,(\text{let})$$

and $\star$ is

$$\dfrac{\dfrac{E, x : \mathsf{int} \vdash (+) : \mathsf{int} \to \mathsf{int} \to \mathsf{int} \qquad E, x : \mathsf{int} \vdash x : \mathsf{int}}{E, x : \mathsf{int} \vdash (+)\ x : \mathsf{int} \to \mathsf{int}}\,(\text{app}) \qquad E, x : \mathsf{int} \vdash x : \mathsf{int}}{E, x : \mathsf{int} \vdash (x + x) : \mathsf{int}}\,(\text{app})$$

(b) *explanation*

$$E \rhd \mathbf{let}\ x = 1\ \mathbf{in}\ \mathsf{p}\ x\ (x + x) : \alpha_0$$

$$\overset{\mathsf{let}}{\Longrightarrow}$$

$$E \rhd 1 : \alpha_1; E, x : \alpha_1 \rhd \mathsf{p}\ x\ (x + x) : \alpha_0$$

$$\overset{\mathsf{con}}{\Longrightarrow}$$

$$\mathsf{int} \approx \alpha_1; E, x : \alpha_1 \rhd \mathsf{p}\ x\ (x + x) : \alpha_0$$

$$\overset{\mathsf{app}}{\Longrightarrow}$$

$$\mathsf{int} \approx \alpha_1; E, x : \alpha_1 \rhd \mathsf{p}\ x : \alpha_2 \to \alpha_0; E, x : \alpha_1 \rhd (x + x) : \alpha_2$$

$$\overset{\mathsf{app}}{\Longrightarrow}$$

$$\mathsf{int} \approx \alpha_1; E, x : \alpha_1 \rhd \mathsf{p} : \alpha_3 \to \alpha_2 \to \alpha_0; E, x : \alpha_1 \rhd x : \alpha_3; E, x : \alpha_1 \rhd (x + x) : \alpha_2$$

$$\overset{\mathsf{con}}{\Longrightarrow}$$

$$\mathsf{int} \approx \alpha_1; \mathsf{int} \to \mathsf{int} \to \mathsf{pair}(\mathsf{int}, \mathsf{int}) \approx \alpha_3 \to \alpha_2 \to \alpha_0;$$
$$E, x : \alpha_1 \rhd x : \alpha_3; E, x : \alpha_1 \rhd (x + x) : \alpha_2$$

$$\overset{\mathsf{con}}{\Longrightarrow}$$

$$\mathsf{int} \approx \alpha_1; \mathsf{int} \to \mathsf{int} \to \mathsf{pair}(\mathsf{int}, \mathsf{int}) \approx \alpha_3 \to \alpha_2 \to \alpha_0;$$
$$\alpha_1 \approx \alpha_3; E, x : \alpha_1 \rhd (x + x) : \alpha_2$$

$$\overset{\mathsf{app}}{\Longrightarrow}$$

$$\mathsf{int} \approx \alpha_1; \mathsf{int} \to \mathsf{int} \to \mathsf{pair}(\mathsf{int}, \mathsf{int}) \approx \alpha_3 \to \alpha_2 \to \alpha_0;$$
$$\alpha_1 \approx \alpha_3; E, x : \alpha_1 \rhd (+)\ x : \alpha_4 \to \alpha_2; E, x : \alpha_1 \rhd x : \alpha_4$$

$$\overset{\mathsf{app}}{\Longrightarrow}$$

$$\mathsf{int} \approx \alpha_1; \mathsf{int} \to \mathsf{int} \to \mathsf{pair}(\mathsf{int}, \mathsf{int}) \approx \alpha_3 \to \alpha_2 \to \alpha_0;$$
$$\alpha_1 \approx \alpha_3; E, x : \alpha_1 \rhd (+) : \alpha_5 \to \alpha_4 \to \alpha_2; E, x : \alpha_1 \rhd x : \alpha_5; E, x : \alpha_1 \rhd x : \alpha_4$$

$$\overset{\mathsf{con}}{\Longrightarrow}$$

$$\mathsf{int} \approx \alpha_1; \mathsf{int} \to \mathsf{int} \to \mathsf{pair}(\mathsf{int}, \mathsf{int}) \approx \alpha_3 \to \alpha_2 \to \alpha_0;$$
$$\alpha_1 \approx \alpha_3; \mathsf{int} \to \mathsf{int} \to \mathsf{int} \approx \alpha_5 \to \alpha_4 \to \alpha_2; E, x : \alpha_1 \rhd x : \alpha_5; E, x : \alpha_1 \rhd x : \alpha_4$$

$$\overset{\mathsf{con}}{\Longrightarrow}$$

$$\mathsf{int} \approx \alpha_1; \mathsf{int} \to \mathsf{int} \to \mathsf{pair}(\mathsf{int}, \mathsf{int}) \approx \alpha_3 \to \alpha_2 \to \alpha_0;$$
$$\alpha_1 \approx \alpha_3; \mathsf{int} \to \mathsf{int} \to \mathsf{int} \approx \alpha_5 \to \alpha_4 \to \alpha_2; \alpha_1 \approx \alpha_5; E, x : \alpha_1 \rhd x : \alpha_4$$

$$\overset{\mathsf{con}}{\Longrightarrow}$$

$$\mathsf{int} \approx \alpha_1; \mathsf{int} \to \mathsf{int} \to \mathsf{pair}(\mathsf{int}, \mathsf{int}) \approx \alpha_3 \to \alpha_2 \to \alpha_0;$$
$$\alpha_1 \approx \alpha_3; \mathsf{int} \to \mathsf{int} \to \mathsf{int} \approx \alpha_5 \to \alpha_4 \to \alpha_2; \alpha_1 \approx \alpha_5; \alpha_1 \approx \alpha_4$$