Introduction to Model Checking

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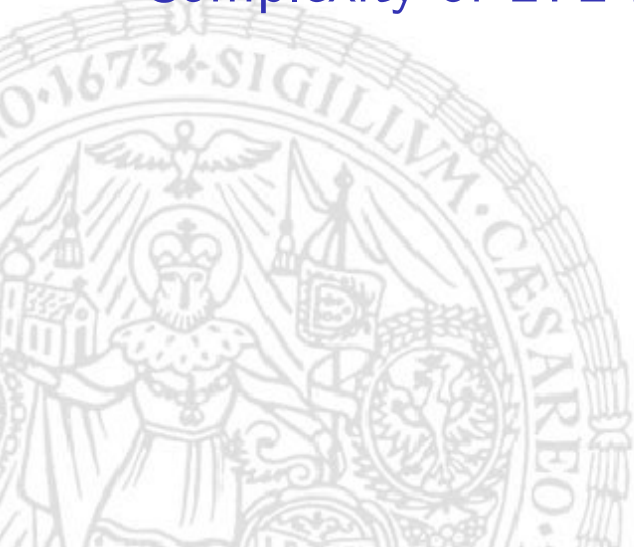
Outline

- Last Lecture
- Complexity of LTL Model Checking



φ-Expansion

a	0 1 1 0 1 0 0 1 1 1 ...
b	0 0 1 1 0 1 0 1 0 1 ...
$\neg b$	1 1 0 0 1 0 1 0 1 0 ...
$\mathsf{X}\, a$	1 1 0 1 0 0 1 1 1 1 ...
$\mathsf{X}\, a \,\mathsf{U}\, b$	1 1 1 1 0 1 1 1 1 1 ...
$\neg b \wedge (\mathsf{X}\, a \,\mathsf{U}\, b)$	1 1 0 0 0 0 1 0 1 0 ...

Idea of LTL to NBA-Translation

- NBA guesses the φ-expansion of w
- ... and checks that guesses are correct
- ... and demands that value for whole formula is 1 for whole word w

Definition (Consistency Checks)

$$
\begin{array}{lll}
\varphi_j = \neg\varphi_{j_1} & \Rightarrow w[i..]^j = 1 \text{ iff} & w[i..]^{j_1} = 0 \\
\varphi_j = \varphi_{j_1} \wedge \varphi_{j_2} & \Rightarrow w[i..]^j = 1 \text{ iff} & w[i..]^{j_1} = 1 \text{ and } w[i..]^{j_2} = 1 \\
\varphi_j = \mathsf{X}\,\varphi_{j_1} & \Rightarrow w[i..]^j = 1 \text{ iff} & w[i+1..]^{j_1} = 1 \\
\varphi_j = \varphi_{j_1}\,\mathsf{U}\,\varphi_{j_2} & \Rightarrow w[i..]^j = 1 \text{ iff} & w[i..]^{j_2} = 1 \text{ or} \\
 & & (w[i..]^{j_1} = 1 \text{ and } w[i+1]^j = 1)
\end{array}
$$

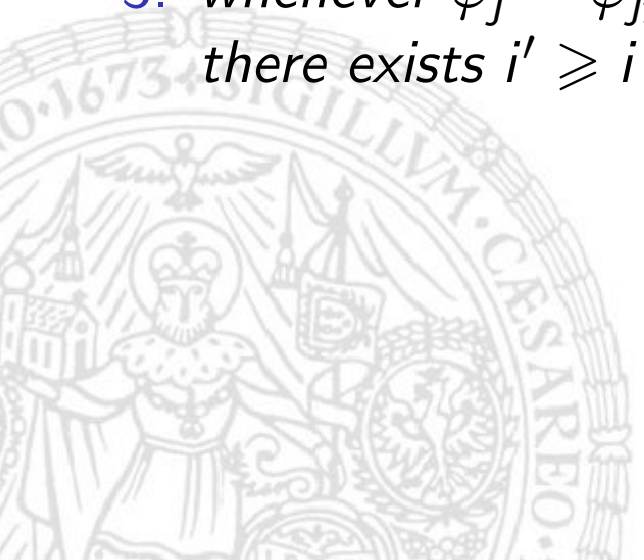
All checks can be performed locally, i.e., column per column!

Consistency Checks and LTL-Models

Lemma

$w \models \varphi$ iff there exists an expansion $v \in (2^m)^\omega$ of w such that

1. *v satisfies the consistency checks*
2. *$v[0..]^m = 1$*
3. *whenever $\varphi_j = \varphi_{j_1} \mathsf{U} \varphi_{j_2}$ and $v[i..]^j = 1$ then there exists $i' \geqslant i$ such that $v[i'..]^{j_2} = 1$*



Translating LTL to GNBA

Definition (GNBA for an LTL formula φ)

Let $cl(\varphi) = a_1, \ldots, a_n, \varphi_{n+1}, \ldots, \varphi_m$ where $\varphi_m = \varphi$.
Define $\mathcal{A}_\varphi = (2^m \uplus \{q_0\}, 2^n, q_0, \delta, F_1, \ldots, F_k)$ where

- $(c_1, \ldots, c_m)^T \in \delta((b_1, \ldots, b_m)^T, (d_1, \ldots, d_n)^T)$ iff
  1. $c_j = d_j$ for all $j \leqslant n$ (expansion)
  2. $(b_1, \ldots, b_m)^T$ $(c_1, \ldots, c_m)^T$ is consistent (consistent expansion)
- $(c_1, \ldots, c_m)^T \in \delta(q_0, (d_1, \ldots, d_n)^T)$ iff
  1. $c_j = d_j$ for all $j \leqslant n$ (expansion)
  2. $(c_1, \ldots, c_m)^T$ is consistent (consistent expansion)
  3. $c_m = 1$ (φ is satisfied)
- if $\varphi_j = \varphi_{j_1} \,\mathsf{U}\, \varphi_{j_2}$ is i-th U-subformula in $cl(\varphi)$ then

$$F_i = \{(b_1, \ldots, b_m)^T \mid b_j = 0 \text{ or } (b_j = 1 \text{ and } b_{j_2} = 1)\}$$

Soundness of Translation

Theorem (Vardi,Wolper)

For every LTL formula φ

$$\mathcal{L}(\varphi) = \mathcal{L}(\mathcal{A}_\varphi)$$

Proof of Lemma.

By induction on φ using the consistency checks. ∎

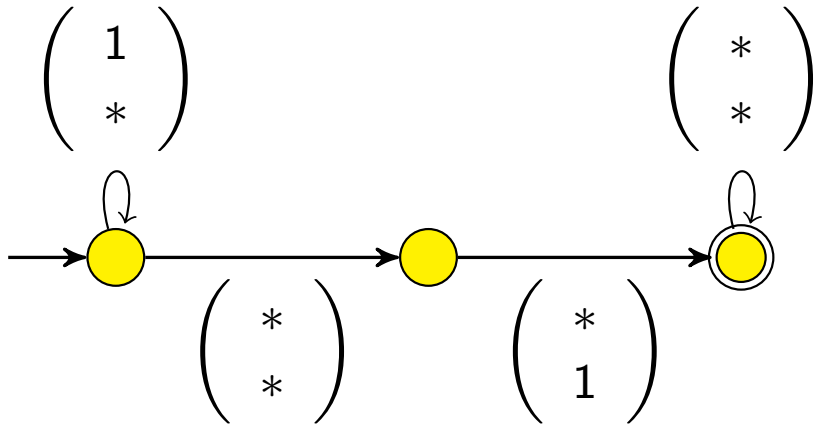
Proof of Theorem.

- Construction of $\mathcal{A}_\varphi$ directly corresponds to requirements 1 and 2 in Lemma
- Remaining difficulty:
  Show that visiting F_i infinitely often is the same as requirement 3 in Lemma for i-th U-subformula $\varphi_j = \varphi_{j_1} \,\mathsf{U}\, \varphi_{j_2}$ ∎

Last Exercises

Construct the NBA for the formula $a \,\mathsf{U}\, \mathsf{X}\, b$

- intuitively by hand



- using the construction of Vardi & Wolper

Complexity of LTL Model Checking

Model Checking: Given TS and φ construct automaton $\mathcal{A}$ such that

$$\mathcal{L}(\mathcal{A}) = (2^n)^\omega \setminus \mathcal{L}(\varphi) \qquad (*)$$

and then check

$$\mathcal{L}(TS \otimes \mathcal{A}) = \varnothing \qquad (**)$$

Two alternatives for $(*)$ available:

- $\mathcal{A} = \overline{\mathcal{A}_\varphi}$ implies $|\mathcal{A}| = \exp(|\mathcal{A}_\varphi|)$ (NBA complementation)
- $\mathcal{A} = \mathcal{A}_{\neg\varphi}$ implies $|\mathcal{A}| = 2 \cdot |\mathcal{A}_\varphi|$ (formula complementation)

Still, $(**)$ has complexity $\mathcal{O}(|TS| \cdot |\mathcal{A}_{\neg\varphi}|) = \mathcal{O}(|TS| \cdot 2^{|\varphi|} \cdot |\varphi|)$

Lower bound

Theorem

There exists a family of LTL formulas φ_n with $|\varphi_n| = \mathcal{O}(poly(n))$ such that every NBA $\mathcal{A}_n$ with $\mathcal{L}(\mathcal{A}_n) = \mathcal{L}(\varphi_n)$ has at least 2^n states.

Proof (1)

Let $AP = \{a\}$ and:

$$\mathcal{L}_n \;=\; \left\{\, b_1 \ldots b_n\, b_1 \ldots b_n\, w \mid b_i \in \{0,1\}, w \in \{0,1\}^\omega \,\right\}, \qquad \text{for } n \geqslant 0$$

It follows $\mathcal{L}_n \;=\; \mathcal{L}(\varphi_n)$ where $\varphi_n \;=\; \bigwedge_{0 \leqslant i < n} (\mathsf{X}^i\, a \Leftrightarrow \mathsf{X}^{n+i}\, a)$

φ_n is an LTL formula of polynomial length: $|\varphi_n| \;\in\; \mathcal{O}(n^2)$

However, any NBA $\mathcal{A}$ with $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_n$ has at least 2^n states

Proof (2)

Is Bad Complexity a Result of Automata Approach? No!

Theorem

The LTL model checking problem $TS \models \varphi$ is coNP-hard.

Proof.

- The Hamiltonian path problem (HPP) is NP-hard.

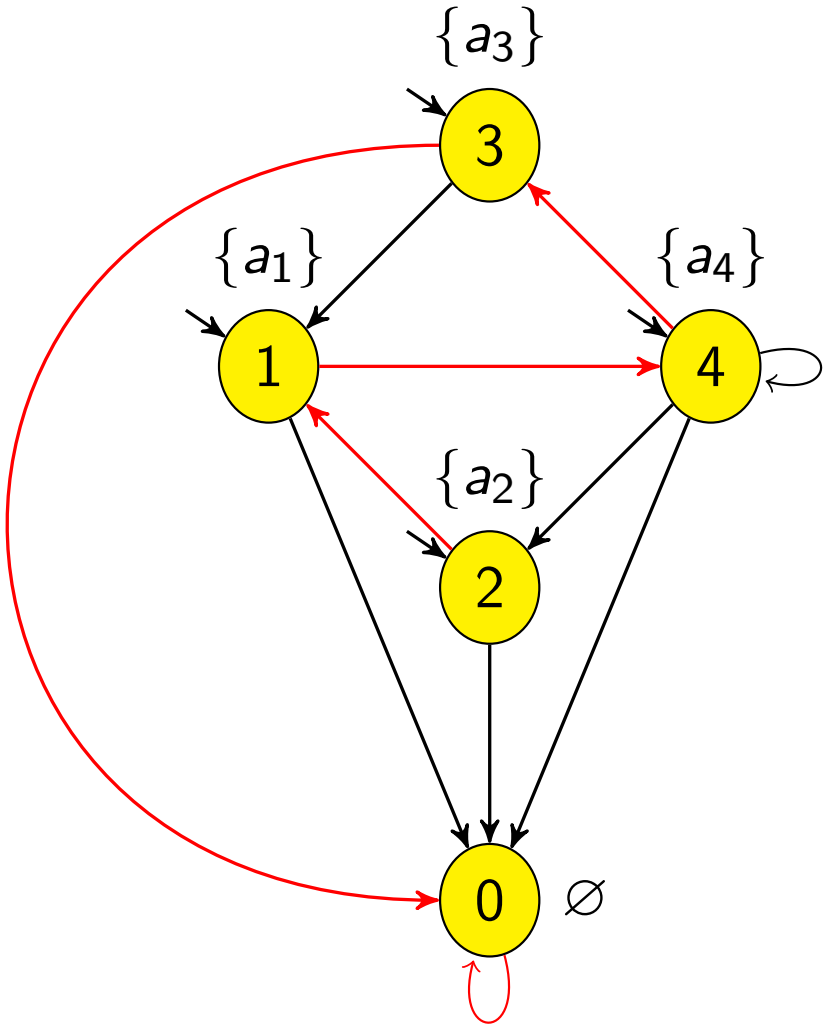
  Given a directed graph $\mathcal{G} = (N, E)$, $E \subseteq N \times N$,
  is there a path which traverses each node exactly once?

- Reduce HPP for graph $\mathcal{G}$ in polynomial time to an LTL model checking problem $(TS_{\mathcal{G}}, \varphi_{\mathcal{G}})$ such that:

  $$\mathcal{G} \text{ has Hamiltonian path} \quad \text{iff} \quad TS_{\mathcal{G}} \not\models \varphi_{\mathcal{G}}$$

  If $N = \{1, \ldots, n\}$ choose $\varphi_{\mathcal{G}}$ and $TS_{\mathcal{G}} = (S, \rightarrow, I, AP, L)$ as follows

Example



Complexity is Even Worse ...

Theorem (Sistla, Clarke)

The LTL model checking problem $TS \models \varphi$ *is PSPACE-hard.*

. . . But There is a Limit

Theorem

The LTL model checking problem $TS \models \varphi$ *is PSPACE-complete.*

Exercises

Show that LTL model checking is coNP-hard by a reduction from SAT.
Reminder:

- The SAT problem is the question whether a propositional formula φ in conjunctive normal form (CNF) is satisfiable. It is the most famous NP-complete problem.
- A formula is in CNF iff it is a conjunction of disjunctions of literals. E.g., the following formula is in CNF:

$$(\neg q \vee \neg p) \wedge p \wedge (r \vee \neg q \vee p)$$

Only give the construction without a formal proof.