Name: ______________________ Points: ____

Matr.-Nr.: ________________

1 First-order logic (15 pts)

Consider the following first-order sentences:

①	$\forall x \; \big(\mathrm{car}(x) \wedge \mathrm{engine}(x, \mathrm{diesel}) \wedge \mathrm{refuel}(x, \mathrm{gas}) \;\rightarrow\; \mathrm{defect}(x)\big)$
②	$\forall x, y \; \big(\mathrm{mechanic}(y) \wedge \mathrm{car}(x) \wedge \mathrm{defect}(x) \rightarrow [\, \mathrm{has}(y, \mathrm{tool}(x)) \rightarrow \mathrm{repair}(y, x) \,]\big)$
③	$\exists x \; \big(\mathrm{car}(x) \wedge \mathrm{blue}(x) \wedge \mathrm{refuel}(x, gas)\big)$
④	$\forall x \; \big(\mathrm{car}(x) \rightarrow (\mathrm{defect}(x) \rightarrow \mathrm{blue}(x)) \wedge (\mathrm{blue}(x) \rightarrow \mathrm{engine}(x, \mathrm{diesel}))\big)$
⑤	$\forall x \; \big(\mathrm{car}(x) \wedge \mathrm{engine}(x, \mathrm{diesel}) \rightarrow \exists y \; (\mathrm{mechanic}(y) \wedge \mathrm{has}(y, \mathrm{tool}(x)))\big)$
⑥	$\exists x \; \big(\mathrm{car}(x) \wedge \mathrm{defect}(x) \wedge \forall y \; (\mathrm{mechanic}(y) \rightarrow \neg\mathrm{repair}(y, x))\big)$

which employ the following constants, functions and predicates:

- constants: *diesel*, *gas* and functions: *tool*(x)
  [first letter suffices, hence one can use $d, g, t(x)$]
- predicates: *car*(x), *engine*(x, y), *refuel*(x, y), *mechanic*(x), *repair*(x, y), *defect*(x), *has*(x, y), *blue*(x) [first 3 letters suffice to identify the predicate]

Thus, the function $t(x)$ returns the tool needed to repair car x. Further, the predicate *eng*(x, y) has to be interpreted as "x has a y engine", the predicate *ref*(x, y) as "x is refuelled with y", the predicate *rep*(x, y) as "x can repair y", and finally the predicate *has*(x, y) as "x has y".

a) Give the natural-language statements which are formalised by the above sentences. [5]

b) Is the conjunction ① ∧ ② ∧ ③ ∧ ④ ∧ ⑤ ∧ ⑥ satisfiable? Prove your answer. [5]

c) Change ② so that the answer to b) changes and prove why it changes. [5]

2 Resolution (5 pts)

Decide for the following set of clauses

$$\{\neg r, \neg s\} \qquad \{q, \neg r, \neg s\} \qquad \{\neg p, q, \neg s\} \qquad \{\neg p, \neg q, \neg s\} \qquad \{\neg q, s\}$$
$$\{p, q, s\} \qquad \{\neg q, r, s\} \qquad \{r, \neg s\} \qquad \{q, r, s\} \qquad \{\neg p, q, \neg r\}$$

whether the empty clause can be derived or not. If the empty clause can be derived, prove it by resolution. Otherwise give a satisfying truth assignment (valuation). [5]

3 Propositional logic (5pt)

Give a set of (introduction and elimination) natural deduction rules for the connective $\leftrightarrow$ which is sound and complete relative to the usual semantics of $\leftrightarrow$, namely $\overline{v}(\phi \leftrightarrow \psi) = \mathsf{T}$ `iff` $\overline{v}(\phi) = \overline{v}(\psi)$. *Hint*: Note the following equivalence: $\phi \leftrightarrow \psi \;\equiv\; (\phi \to \psi) \wedge (\psi \to \phi)$.

[5]

4 Theoretical Questions (15 pts)

a) Give the definitions of *Prenex form*, *Skolem form*, *Herbrand universe*, *Herbrand interpretation* and *Herbrand model*. [3]

b) State the *Löwenheim-Skolem* theorem and the *main lemma* used to prove it. Prove the Löwenheim-Skolem in terms of this main lemma. [5]

c) Outline the proof of the above main lemma. Give all the main steps of its proof. [5]

d) State the *compactness theorem*. (No proof is required, but a correct statement!) [2]