1 De-Formalisation

a)

Natural-language statements:

① If a car has a diesel engine and is refueled with gas then it gets defect.

② A mechanic does repair a defect car if she/he has the necessary tool for that car.

③ There is a blue car which is refueled with gas.

④ All defect cars are blue and all blue cars have diesel engine.

⑤ For every car with diesel engine there is a mechanic which has the necessary tool for that car.

⑥ There is a defect car which cannot be repaired by any mechanic.

b)

The conjunction is unsatisfiable:

> The system is inconsistent (it has no model) since by ① ∧ ② ∧ ④ ∧ ⑤ one proves that for any defect car (which must be blue and with diesel engine) there is a mechanic which can repair it, in contradiction to ⑥. Alternatively, ② ∧ ④ ∧ ⑤ ∧ ⑥ can straightforwardly be proved to be unsatisfiable. *Note that there are models of ② ∧ ⑥, even though they contradict each-other in certain other models. Fun exercise: build such models (in both categories)!*

c)

The conjunction becomes satisfiable by reverting the rightmost implication in ②:

> Hence we change ② to $\forall x, y\ \big(\mathrm{mechanic}(y) \wedge \mathrm{car}(x) \wedge \mathrm{defect}(x) \rightarrow (\neg\mathrm{has}(y, \mathrm{tool}(x)) \rightarrow \neg\mathrm{repair}(y, x))\big)$ which in natural language reads *"A mechanic cannot repair a defect car if she/he does not have the necessary tool for that car."* and then the conjunction becomes satisfiable with the model $(\mathcal{M}, \emptyset)$ formally defined here below. The idea is that the car x from clause ③ can be proved to be defect (since it has a diesel engine but is refueled with gas) and no mechanic can repair it in the model $\mathcal{M}$, and this despite the fact that there is a mechanic which has the tool needed to repair it (but there may be some other reason why this mechanic cannot repair this car, e.g., he gets ill and cannot work). The simplest $\mathcal{M}$ has a 4-element universe $A = \{\mathtt{m}, \mathtt{c}, \mathtt{g}, \mathtt{d}\}$, with $\mathrm{diesel}^{\mathcal{M}} = \mathtt{d}$, $\mathrm{gas}^{\mathcal{M}} = \mathtt{g}$, $\mathrm{mechanic}^{\mathcal{M}} = \{\mathtt{m}\}$, $\mathrm{car}^{\mathcal{M}} = \mathrm{defect}^{\mathcal{M}} = \mathrm{blue}^{\mathcal{M}} = \{\mathtt{c}\}$, $\mathrm{engine}^{\mathcal{M}} = \{(\mathtt{c}, \mathtt{d})\}$, $\mathrm{has}^{\mathcal{M}} = \{(\mathtt{m}, \mathtt{t}(\mathtt{c}))\}$ (here $\mathrm{tool}^{\mathcal{M}} = \mathtt{t} : A \mapsto A$ is, e.g., the identity) and yet $\mathrm{repair}^{\mathcal{M}} = \emptyset$.

3 Propositional logic (5 pts)

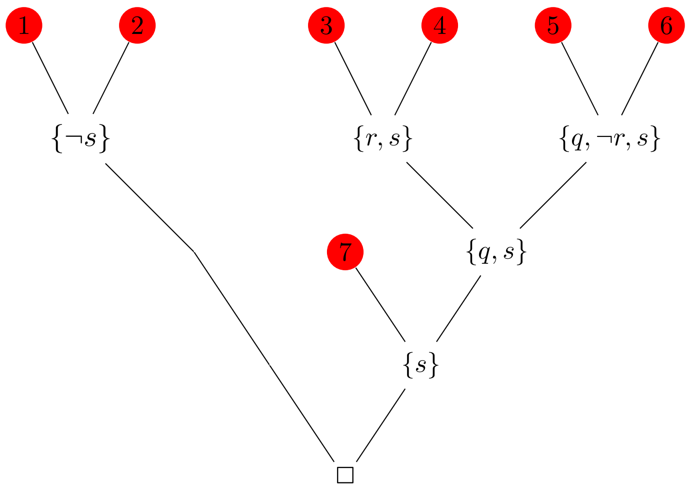
$$\dfrac{\begin{matrix}\phi & & \psi \\ \vdots & & \vdots \\ \psi & & \phi\end{matrix}}{\phi \leftrightarrow \psi}\ \leftrightarrow_i \qquad \dfrac{\phi \quad \phi \leftrightarrow \psi}{\psi}\ \leftrightarrow_{e_1} \qquad \dfrac{\psi \quad \phi \leftrightarrow \psi}{\phi}\ \leftrightarrow_{e_2}$$

2 Resolution

Sufficient to consider $S = \{ \{r, \neg s\}$ 1 $, \{\neg r, \neg s\}$ 2 $, \{\neg q, r, s\}$ 3 $, \{q, r, s\}$ 4 ,

$\{p, q, s\}$ 5 $, \{\neg p, q, \neg r\}$ 6 $, \{\neg q, s\}$ 7 $\}$

We can build the following resolution derivation:



4 Theoretical Questions (15 pts)

Prenex form and Skolem form: a)

A formula ϕ is in *prenex form* if it is of shape $Q_1 x_1 \ldots Q_n x_n\ \psi$, where $Q_i \in \{\forall, \exists\}$ and ψ is quantifier-free. If moreover $Q_1 = \ldots = Q_n = \forall$, then we say that ϕ is in *Skolem form*.

Herbrand universe, Herbrand interpretation and Herbrand model:

Let ϕ be a formula in Skolem form and $\mathcal{F}$ be the minimal signature which contains all constant and function symbols occuring in ϕ, plus one extra constant. The *Herbrand universe* H_ϕ of ϕ is the set of all variable-free terms that can be built over $\mathcal{F}$. Note that $H_\phi \neq \emptyset$.

An interpretation $\mathcal{H}$ (over signature $\mathcal{F}$) with universe H_ϕ is a *Herbrand interpretation* of ϕ if $f^{\mathcal{H}}(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$ for all f occurring in ϕ (of arity $n \geq 0$).

We further say that $(\mathcal{H}, l)$ is a *Herbrand model* of ϕ if additionally $\mathcal{H} \models_l \phi$.

Loewenheim-Skolem theorem and its main lemma: b)&c)

Lemma: A sentence ϕ in Skolem form is satisfiable `iff` ϕ has a Herbrand model.

Theorem: If a sentence ϕ has a model with an infinite universe then it also has a model with a countable universe.

Proof: Let ψ be a Skolem form for ϕ. Then ϕ has a model `iff` ψ has a Herbrand model. Notice that H_ψ is countable. Q.e.d.

d)

Statement of the Compactness Theorem (2 pts):

A set of sentences Γ is satisfiable `iff` all its finite subsets $\Delta \subseteq \Gamma$ are satisfiable.