Solutions

1. Consider the lambda-term $t = (\lambda x\ y.x\ (x\ y))\ (\lambda x.s\ x)\ z$.

[10] (a) Reduce t stepwise to normal form, using the leftmost innermost strategy.

Solution.

$$\begin{aligned}
\underline{(\lambda x\ y.x\ (x\ y))\ (\lambda x.s\ x)}\ z &\to_\beta (\lambda y.(\lambda x.s\ x)\ (\underline{(\lambda x.s\ x)\ y}))\ z \\
&\to_\beta (\lambda y.\underline{(\lambda x.s\ x)\ (s\ y)})\ z \\
&\to_\beta \underline{(\lambda y.s\ (s\ y))\ z} \\
&\to_\beta s\ (s\ z)
\end{aligned}$$

[10] (b) Reduce t stepwise to normal form, using the leftmost outermost strategy.

Solution.

$$\begin{aligned}
\underline{(\lambda x\ y.x\ (x\ y))\ (\lambda x.s\ x)}\ z &\to_\beta \underline{(\lambda y.(\lambda x.s\ x)\ ((\lambda x.s\ x)\ y))\ z} \\
&\to_\beta \underline{(\lambda x.s\ x)\ ((\lambda x.s\ x)\ z)} \\
&\to_\beta s\ (\underline{(\lambda x.s\ x)\ z}) \\
&\to_\beta s\ (s\ z)
\end{aligned}$$

[20] **2.** Consider the type

```
type nat = Z | S of nat
```

together with the functions

```
let rec ( + ) x y = match x with Z -> y | S x -> S(x + y)
let rec ( * ) x y = match x with Z -> Z | S x -> y + x * y
```

Prove by induction that $a * (b + c) = a * b + a * c$ for all `nat`s a, b, and c. You may use associativity and commutativity of `+`, i.e.,

$$a + (b + c) = (a + b) + c \qquad (\star)$$

$$a + b = b + a \qquad (\star\star)$$

for all `nat`s a, b, and c.

Solution.

Base Case ($a = \mathtt{Z}$). The base case concludes by the derivation

$$\begin{aligned}
\mathtt{Z} * (b + c) &= \mathtt{Z} && \text{(def. of } *) \\
&= \mathtt{Z} + \mathtt{Z} && \text{(def. of } +) \\
&= \mathtt{Z} * b + \mathtt{Z} && \text{(def. of } *) \\
&= \mathtt{Z} * b + \mathtt{Z} * c && \text{(def. of } *)
\end{aligned}$$

Step Case ($a = \mathtt{S}\ a'$). By IH we may assume $a' * (b + c) = a' * b + a' * c$. The step case concludes by the derivation

$$\begin{aligned} a * (b + c) &= \mathtt{S}\ a' * (b + c) \\ &= (b + c) + a' * (b + c) && \text{(def. of } *) \\ &= (b + c) + (a' * b + a' * c) && \text{(by IH)} \\ &= (b + a' * b) + (c + a' * c) && \text{(by } (\star) \text{ and } (\star\star)) \\ &= \mathtt{S}\ a' * b + \mathtt{S}\ a' * c && \text{(def. of } *) \\ &= a * b + a * c \end{aligned}$$

3. Consider the OCaml function `initial`, defined by:

```
let rec initial = function x::y::xs -> x :: initial(y::xs) | _ -> []
```

[10] (a) Implement a tail-recursive variant of `initial`. You may assume to have a tail-recursive function `rev` (reversing a list) at hand.

Solution.

```
let initial xs =
  let rec initial acc = function x::y::xs -> initial (x::acc) (y::xs)
                               | _        -> rev acc
  in
  initial [] xs
```

[10] (b) Implement the function `even_odd` that splits a list into two lists, where the first contains all elements having even indices and the second all those with odd indices.

Solution.

```
let rec even_odd = function
  | x::y::xs -> let (e,o) = even_odd xs in (x::e,y::o)
  | xs       -> (xs,[])
```

4. Consider the λ-term $t = (\lambda x\ y.x\ (x\ y))\ (\lambda x.s\ x)\ z$.

[5] (a) Is t in normal form? (Justify your answer.)

Solution. No, as can be seen by the reductions from **1.**(a) and **1.**(b).

[5] (b) Give the set $\mathcal{FV}\mathsf{ar}(t)$ of free variables of t.

Solution. $\mathcal{FV}\mathsf{ar}(t) = \{s, z\}$

[5] (c) Give the set $\mathcal{BV}\mathsf{ar}(t)$ of bound variables of t.

Solution. $\mathcal{BV}\mathsf{ar}(t) = \{x, y\}$

[5] (d) Give the set $\mathcal{S}\mathsf{ub}(t)$ of all subterms of t.

Solution. $\mathcal{S}\mathsf{ub}(t) = \{t, \lambda x\ y.x\ (x\ y), \lambda y.x\ (x\ y), \lambda x.s\ x, x\ (x\ y), x\ y, s\ x, s, x, y, z\}$

[10] **5.** (a) Transform the type inference problem $E \rhd (\lambda x\ y.x\ (x\ y))\ \mathsf{s}\ \mathsf{0} : \alpha_0$, using the environment $E = \{\mathsf{s} : \mathsf{int} \to \mathsf{int}, \mathsf{0} : \mathsf{int}\}$, into a unification problem.

Solution.

$$E \rhd (\lambda x\ y.x\ (x\ y))\ \mathsf{s}\ \mathsf{0} : \alpha_0$$
$$\stackrel{\mathsf{app}}{\Rightarrow}$$
$$E \rhd (\lambda x\ y.x\ (x\ y))\ \mathsf{s} : \alpha_1 \to \alpha_0; E \rhd \mathsf{0} : \alpha_1$$
$$\stackrel{\mathsf{app}}{\Rightarrow}$$
$$E \rhd \lambda x\ y.x\ (x\ y) : \alpha_2 \to \alpha_1 \to \alpha_0; E \rhd \mathsf{s} : \alpha_2; E \rhd \mathsf{0} : \alpha_1$$
$$\stackrel{\mathsf{abs}}{\Rightarrow}$$
$$E, x : \alpha_3 \rhd \lambda y.x\ (x\ y) : \alpha_4; \alpha_2 \to \alpha_1 \to \alpha_0 \approx \alpha_3 \to \alpha_4; E \rhd \mathsf{s} : \alpha_2; E \rhd \mathsf{0} : \alpha_1$$
$$\stackrel{\mathsf{abs}}{\Rightarrow}$$
$$E, x : \alpha_3, y : \alpha_5 \rhd x\ (x\ y) : \alpha_6; \alpha_4 \approx \alpha_5 \to \alpha_6; \alpha_2 \to \alpha_1 \to \alpha_0 \approx \alpha_3 \to \alpha_4; E \rhd \mathsf{s} : \alpha_2; E \rhd \mathsf{0} : \alpha_1$$
$$\stackrel{\mathsf{app}}{\Rightarrow}$$
$$E, x : \alpha_3, y : \alpha_5 \rhd x : \alpha_7 \to \alpha_6; E, x : \alpha_3, y : \alpha_5 \rhd x\ y : \alpha_7; \alpha_4 \approx \alpha_5 \to \alpha_6;$$
$$\alpha_2 \to \alpha_1 \to \alpha_0 \approx \alpha_3 \to \alpha_4; E \rhd \mathsf{s} : \alpha_2; E \rhd \mathsf{0} : \alpha_1$$
$$\stackrel{\mathsf{con}}{\Rightarrow}$$
$$\alpha_3 \approx \alpha_7 \to \alpha_6; E, x : \alpha_3, y : \alpha_5 \rhd x\ y : \alpha_7; \alpha_4 \approx \alpha_5 \to \alpha_6;$$
$$\alpha_2 \to \alpha_1 \to \alpha_0 \approx \alpha_3 \to \alpha_4; E \rhd \mathsf{s} : \alpha_2; E \rhd \mathsf{0} : \alpha_1$$
$$\stackrel{\mathsf{app}}{\Rightarrow}$$
$$\alpha_3 \approx \alpha_7 \to \alpha_6; E, x : \alpha_3, y : \alpha_5 \rhd x : \alpha_8 \to \alpha_7; E, x : \alpha_3, y : \alpha_5 \rhd y : \alpha_8; \alpha_4 \approx \alpha_5 \to \alpha_6;$$
$$\alpha_2 \to \alpha_1 \to \alpha_0 \approx \alpha_3 \to \alpha_4; E \rhd \mathsf{s} : \alpha_2; E \rhd \mathsf{0} : \alpha_1$$
$$\stackrel{\mathsf{con}}{\Rightarrow}$$
$$\alpha_3 \approx \alpha_7 \to \alpha_6; \alpha_3 \approx \alpha_8 \to \alpha_7; E, x : \alpha_3, y : \alpha_5 \rhd y : \alpha_8; \alpha_4 \approx \alpha_5 \to \alpha_6;$$
$$\alpha_2 \to \alpha_1 \to \alpha_0 \approx \alpha_3 \to \alpha_4; E \rhd \mathsf{s} : \alpha_2; E \rhd \mathsf{0} : \alpha_1$$
$$\stackrel{\mathsf{con}}{\Rightarrow}$$
$$\alpha_3 \approx \alpha_7 \to \alpha_6; \alpha_3 \approx \alpha_8 \to \alpha_7; \alpha_5 \approx \alpha_8; \alpha_4 \approx \alpha_5 \to \alpha_6;$$
$$\alpha_2 \to \alpha_1 \to \alpha_0 \approx \alpha_3 \to \alpha_4; E \rhd \mathsf{s} : \alpha_2; E \rhd \mathsf{0} : \alpha_1$$
$$\stackrel{\mathsf{con}}{\Rightarrow}$$
$$\alpha_3 \approx \alpha_7 \to \alpha_6; \alpha_3 \approx \alpha_8 \to \alpha_7; \alpha_5 \approx \alpha_8; \alpha_4 \approx \alpha_5 \to \alpha_6;$$
$$\alpha_2 \to \alpha_1 \to \alpha_0 \approx \alpha_3 \to \alpha_4; \mathsf{int} \to \mathsf{int} \approx \alpha_2; E \rhd \mathsf{0} : \alpha_1$$
$$\stackrel{\mathsf{con}}{\Rightarrow}$$
$$\alpha_3 \approx \alpha_7 \to \alpha_6; \alpha_3 \approx \alpha_8 \to \alpha_7; \alpha_5 \approx \alpha_8; \alpha_4 \approx \alpha_5 \to \alpha_6;$$
$$\alpha_2 \to \alpha_1 \to \alpha_0 \approx \alpha_3 \to \alpha_4; \mathsf{int} \to \mathsf{int} \approx \alpha_2; \mathsf{int} \approx \alpha_1$$

[10] (b) Solve the following unification problem (if possible).

$$\begin{aligned}
\alpha_3 &\approx \alpha_7 \to \alpha_6 & \alpha_2 \to \alpha_1 \to \alpha_0 &\approx \alpha_3 \to \alpha_4\\
\alpha_3 &\approx \alpha_8 \to \alpha_7 & \mathsf{int} \to \mathsf{int} &\approx \alpha_2\\
\alpha_5 &\approx \alpha_8 & \mathsf{int} &\approx \alpha_1\\
\alpha_4 &\approx \alpha_5 \to \alpha_6 & &
\end{aligned}$$

Solution.

$$\alpha_3 \approx \alpha_7 \to \alpha_6; \alpha_3 \approx \alpha_8 \to \alpha_7; \alpha_5 \approx \alpha_8; \alpha_4 \approx \alpha_5 \to \alpha_6;$$
$$\alpha_2 \to \alpha_1 \to \alpha_0 \approx \alpha_3 \to \alpha_4; \mathsf{int} \to \mathsf{int} \approx \alpha_2; \mathsf{int} \approx \alpha_1$$
$$\Rightarrow^{(\mathsf{v}_1)}_{\{\alpha_3/\alpha_7\to\alpha_6\}}$$
$$\alpha_7 \to \alpha_6 \approx \alpha_8 \to \alpha_7; \alpha_5 \approx \alpha_8; \alpha_4 \approx \alpha_5 \to \alpha_6;$$
$$\alpha_2 \to \alpha_1 \to \alpha_0 \approx (\alpha_7 \to \alpha_6) \to \alpha_4; \mathsf{int} \to \mathsf{int} \approx \alpha_2; \mathsf{int} \approx \alpha_1$$
$$\Rightarrow^{(\mathsf{d}_2)}_{\iota}$$
$$\alpha_7 \approx \alpha_8; \alpha_6 \approx \alpha_7; \alpha_7 \to \alpha_6 \approx \alpha_8 \to \alpha_7; \alpha_5 \approx \alpha_8; \alpha_4 \approx \alpha_5 \to \alpha_6;$$
$$\alpha_2 \to \alpha_1 \to \alpha_0 \approx (\alpha_7 \to \alpha_6) \to \alpha_4; \mathsf{int} \to \mathsf{int} \approx \alpha_2; \mathsf{int} \approx \alpha_1$$
$$\Rightarrow^{(\mathsf{v}_1)}_{\{\alpha_7/\alpha_8\}}$$
$$\alpha_6 \approx \alpha_8; \alpha_8 \to \alpha_6 \approx \alpha_8 \to \alpha_8; \alpha_5 \approx \alpha_8; \alpha_4 \approx \alpha_5 \to \alpha_6;$$
$$\alpha_2 \to \alpha_1 \to \alpha_0 \approx (\alpha_8 \to \alpha_6) \to \alpha_4; \mathsf{int} \to \mathsf{int} \approx \alpha_2; \mathsf{int} \approx \alpha_1$$
$$\Rightarrow^{(\mathsf{v}_1)}_{\{\alpha_6/\alpha_8\}}$$
$$\alpha_8 \to \alpha_8 \approx \alpha_8 \to \alpha_8; \alpha_5 \approx \alpha_8; \alpha_4 \approx \alpha_5 \to \alpha_8;$$
$$\alpha_2 \to \alpha_1 \to \alpha_0 \approx (\alpha_8 \to \alpha_8) \to \alpha_4; \mathsf{int} \to \mathsf{int} \approx \alpha_2; \mathsf{int} \approx \alpha_1$$
$$\Rightarrow^{(\mathsf{t})}_{\iota}$$
$$\alpha_5 \approx \alpha_8; \alpha_4 \approx \alpha_5 \to \alpha_8;$$
$$\alpha_2 \to \alpha_1 \to \alpha_0 \approx (\alpha_8 \to \alpha_8) \to \alpha_4; \mathsf{int} \to \mathsf{int} \approx \alpha_2; \mathsf{int} \approx \alpha_1$$
$$\Rightarrow^{(\mathsf{v}_1)}_{\{\alpha_5/\alpha_8\}}$$
$$\alpha_4 \approx \alpha_8 \to \alpha_8;$$
$$\alpha_2 \to \alpha_1 \to \alpha_0 \approx (\alpha_8 \to \alpha_8) \to \alpha_4; \mathsf{int} \to \mathsf{int} \approx \alpha_2; \mathsf{int} \approx \alpha_1$$
$$\Rightarrow^{(\mathsf{v}_1)}_{\{\alpha_4/\alpha_8\to\alpha_8\}}$$
$$\alpha_2 \to \alpha_1 \to \alpha_0 \approx (\alpha_8 \to \alpha_8) \to \alpha_8 \to \alpha_8; \mathsf{int} \to \mathsf{int} \approx \alpha_2; \mathsf{int} \approx \alpha_1$$
$$\Rightarrow^{(\mathsf{d}_1)}_{\iota}$$
$$\alpha_2 \approx \alpha_8 \to \alpha_8; \alpha_1 \to \alpha_0 \approx \alpha_8 \to \alpha_8; \mathsf{int} \to \mathsf{int} \approx \alpha_2; \mathsf{int} \approx \alpha_1$$

$$\Rightarrow^{(\mathsf{v}_1)}_{\{\alpha_2/\alpha_8\to\alpha_8\}}$$

$$\alpha_1 \to \alpha_0 \approx \alpha_8 \to \alpha_8; \mathsf{int} \to \mathsf{int} \approx \alpha_8 \to \alpha_8; \mathsf{int} \approx \alpha_1$$

$$\Rightarrow^{(\mathsf{d}_2)}_{\iota}$$

$$\alpha_1 \approx \alpha_8; \alpha_0 \approx \alpha_8; \mathsf{int} \to \mathsf{int} \approx \alpha_8 \to \alpha_8; \mathsf{int} \approx \alpha_1$$

$$\Rightarrow^{(\mathsf{v}_1)}_{\{\alpha_1/\alpha_8\}}$$

$$\alpha_0 \approx \alpha_8; \mathsf{int} \to \mathsf{int} \approx \alpha_8 \to \alpha_8; \mathsf{int} \approx \alpha_8$$

$$\Rightarrow^{(\mathsf{v}_1)}_{\{\alpha_0/\alpha_8\}}$$

$$\mathsf{int} \to \mathsf{int} \approx \alpha_8 \to \alpha_8; \mathsf{int} \approx \alpha_8$$

$$\Rightarrow^{(\mathsf{d}_2)}_{\iota}$$

$$\mathsf{int} \approx \alpha_8; \mathsf{int} \approx \alpha_8; \mathsf{int} \approx \alpha_8$$

$$\Rightarrow^{(\mathsf{v}_2)}_{\{\alpha_8/\mathsf{int}\}}$$

$$\mathsf{int} \approx \mathsf{int}; \mathsf{int} \approx \mathsf{int}$$

$$\Rightarrow^{(\mathsf{t})}_{\iota}$$

$$\mathsf{int} \approx \mathsf{int}$$

$$\Rightarrow^{(\mathsf{t})}_{\iota}$$

$$\Box$$

This leads to the solution

$$\{\alpha_0/\mathsf{int}, \alpha_1/\mathsf{int}, \alpha_2/\mathsf{int} \to \mathsf{int}, \alpha_3/\mathsf{int} \to \mathsf{int}, \alpha_4/\mathsf{int} \to \mathsf{int}, \alpha_5/\mathsf{int}, \alpha_6/\mathsf{int}, \alpha_7/\mathsf{int}, \alpha_8/\mathsf{int}\}$$